A

TREATISE

ON

ALGEBRA.

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PREFACE.

The stereotype plates of my Treatise on Algebra having become so much worn in the printing of more than 60,000 copies that it had become necessary to cast them aside, I decided to improve the opportunity to make a thorough revision of the work. I therefore solicited criticisms from several college professors who had had much experience in the use of this book, and in reply have received numerous suggestions. The book has been almost entirely rewritten, nearly every page of it having been given to the printer in manuscript. The general plan of the original work has not been materially altered, but the changes of arrangement and of execution are numerous. In the former editions, in place of abstruse demonstrations, I sometimes employed numerical illustrations, or deductions from particular examples. In the present edition such methods have been discarded, and I have aimed to demonstrate with conciseness and elegance every principle which is propounded.

This book therefore aims to exhibit in logical order all those principles of Algebra which are most important as a preparation for the subsequent branches of a college course of mathematics. I have retained, with but slight alteration, a feature which was made prominent in the former editions, that of stating each problem twice: first as a restricted numerical problem, and then in a more general form, aiming thereby to lead the student to cultivate the faculty of generalization. At the same time I have very much increased the number of examples incorporated with each chapter of the book, and at the close have given a large collection of examples, to which the teacher may resort whenever occasion may require.

The proofs of the work have all been examined by Prof. H. A. Newton, to whom I am indebted for numerous and important suggestions.

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ALGEBRA.

CHAPTER I.

DEFINITIONS AND NOTATION.

1. *Quantity* is any thing that can be increased or diminished, and that can be measured.

A line, a surface, a solid, a weight, etc., are quantities; but the operations of the mind, such as memory, imagination, judgment, etc., are *not* quantities.

A quantity is measured by finding how many times it contains some other quantity of the same kind taken as a standard. The assumed standard is called the *unit of measure.*

2. *Mathematics* is the science of quantity, or the science which treats of the properties and relations of quantities. It employs a variety of symbols to express the values and relations of quantities, and the operations to be performed upon these quantities, or upon the numbers which represent these quantities.

3. Mathematics is divided into *pure* and *mixed.* Pure mathematics comprehends all inquiries into the relations of magnitude in the abstract, and without reference to material bodies. It embraces numerous subdivisions, such as Arithmetic, Algebra, Geometry, etc.

In the mixed mathematics, these abstract principles are applied to various questions which occur in nature. Thus, in Surveying, the abstract principles of Geometry are applied to the measurement of land; in Navigation, the same principles are applied to the determination of a ship's place at sea; in Optics, they are employed to investigate the properties of light; and in Astronomy, to determine the distances of the heavenly bodies.

4. *Algebra* is that branch of mathematics in which quantities are represented by letters, and their relations to each other, as well as the operations to be performed upon them, are indicated by signs or symbols. The object of algebraic notation is to abridge and generalize the reasoning employed in the solution of all questions relating to numbers. Algebra may therefore be called a species of *Universal Arithmetic.*

5. The symbols employed in Algebra may be divided into *three classes:*

1st. Symbols which denote quantities.

2d. Symbols which indicate operations to be performed upon quantities.

3d. Symbols which indicate the relations subsisting between different quantities, with respect to their magnitudes, etc.

Symbols which denote Quantities.

6. In order to generalize our reasoning respecting numbers, we represent them by *letters*, as a, b, c, or x, y, z, etc., and these may represent any numbers whatever. The quantities thus represented may be either *known* quantities—that is, quantities whose values are given; or *unknown* quantities—that is, quantities whose values are to be determined.

Known quantities are generally represented by the *first* letters of the alphabet, as a, b, c, d, etc., and unknown quantities by the *last* letters of the alphabet, as x, y, z, u, etc. This, however, is not a necessary rule, and is not always observed.

7. Sometimes several quantities are represented by a single letter, repeated with different *accents*, as a', a'', a''', a'''', etc., which are read a prime, a second, a third, etc.; or by a letter repeated with different subscript figures, as a_1, a_2, a_3, a_4, etc., which may be read a one sub, a two sub, a three sub, etc. All these symbols represent different quantities, but the accents or numerals are employed to indicate some important relation between the quantities represented.

8. Sometimes quantities are represented by the *initial letters* of their names. Thus s may represent *sum;* d, *difference* or *diameter;* r, *radius* or *ratio;* c, *circumference;* h, *height*, etc. All these letters may be used with accents. Thus, in a problem relating to two circles, d may represent the diameter of one circle, and d' the diameter of the other; c the circumference of one, and c' the circumference of the other, etc.

Symbols which indicate Operations.

9. The *sign of addition* is an erect cross, $+$, called *plus*, and when placed between two quantities it indicates that the second is to be added to the first. Thus, $5+3$ indicates that we must add 3 to the number 5, in which case the result is 8. We also make use of the same sign to connect several numbers together. Thus, $7+5+9$ indicates that to the number 7 we must add 5 and also 9, which make 21. So, also, $8+5+13+11+1+3+10$ is equal to 51.

The expression $a+b$ indicates the sum of two numbers, which we represent by a and b. In the same manner, $m+n+x+y$ indicates the sum of the numbers represented by these four letters. If we knew, therefore, the numbers represented by the letters, we could easily find by arithmetic the value of such expressions.

10. The *sign of subtraction* is a short horizontal line, $-$, called *minus*. When placed between two quantities, it indicates that the second is to be subtracted from the first. Thus, $8-5$ indicates that the number 5 is to be taken from the number 8, which leaves a remainder of 3. In like manner, $12-7$ is equal to 5, etc.

Sometimes we may have several numbers to subtract from a single one. Thus, $16-5-4$ indicates that 5 is to be subtracted from 16, and this remainder is to be further diminished by 4, leaving 7 for the result. In the same manner, $50-1-5-3-9-7$ is equal to 25.

The expression $a-b$ indicates that the number designated by a is to be diminished by the number designated by b.

11. The *double sign* $\pm$ is sometimes written before a quantity to indicate that in certain cases it is to be added, and in others it is to be subtracted. Thus, $b \pm c$ is read *b plus or minus c*, and denotes either the sum or the difference of these two quantities.

12. The *sign of multiplication* is an inclined cross, $\times$. When placed between two quantities, it indicates that the first is to be multiplied by the second. Thus, 3×5 indicates that 3 is to be multiplied by 5, making 15. In like manner, $a \times b$ indicates that a is to be multiplied by b; and $a \times b \times c$ indicates the continued product of the numbers designated by a, b, and c, and so on for any number of quantities.

Multiplication is also frequently indicated by placing a point between the successive letters. Thus, $a.b.c.d$ signifies the same thing as $a \times b \times c \times d$.

Generally, however, when numbers are represented by letters, their multiplication is indicated by writing them in succession without any intervening sign. Thus, abc signifies the same as $a \times b \times c$, or $a.b.c$.

The notation $a.b$ or ab is seldom employed except when the numbers are designated by letters. If, for example, we attempt to represent in this manner the product of the numbers 5 and 6, 5.6 might be confounded with $5\frac{6}{10}$; and 56 would be read *fifty-six*, instead of *five times six*.

The multiplication of numbers may, however, be denoted by placing a point between them in cases where *no ambiguity* can arise from the use of this symbol. Thus, 1.2.3.4.5 is sometimes used to represent the continued product of the numbers 1, 2, 3, 4, 5.

13. When two or more quantities are multiplied together, each of them is called a *factor*. Thus, in the expression 7×5, 7 is a factor, and so is 5. In the product abc there are three factors, a, b, c.

When a quantity is represented by a letter, it is called a *literal* factor. When it is represented by a figure or figures, it is called a *numerical* factor. Thus, in the expression $5ab$, 5 is a numerical factor, while a and b are literal factors.

14. The *sign of division* is a short horizontal line with a point above and one below, $\div$. When placed between two quantities, it indicates that the first is to be divided by the second.

Thus, $24\div 6$ indicates that 24 is to be divided by 6, making 4. So, also, $a\div b$ indicates that a is to be divided by b.

Generally, however, the division of two numbers is indicated by writing the divisor under the dividend, and drawing a line between them. Thus, $24\div 6$ and $a\div b$ are usually written $\frac{24}{6}$ and $\frac{a}{b}$.

15. The products formed by the successive multiplication of the same number by itself, are called the *powers* of that number.

Thus, $2\times 2=4$, the second power or square of 2.
$2\times 2\times 2=8$, the third power or cube of 2.
$2\times 2\times 2\times 2=16$, the fourth power of 2, etc.
So, also, $3\times 3=9$, the second power of 3.
$3\times 3\times 3=27$, the third power of 3, etc.
Also, $a\times a=aa$, the second power of a.
$a\times a\times a=aaa$, the third power of a, etc.

In general, any power of a quantity is designated by the number of equal factors which form the product.

16. The *sign of involution* is a number written above a quantity, at the right hand, to indicate how many times the quantity is to be taken as a factor.

A *root* of a quantity is a factor which, multiplied by itself a certain number of times, will produce the given quantity.

The figure which indicates how many times the root or factor is taken, is called the *exponent* of the power.

Thus, instead of aa, we write a^2, where 2 is the exponent of the power; instead of aaa, we write a^3, where 3 is the exponent of the power; instead of *aaaaa*, we write a^5, where 5 is the exponent of the power, etc.

When no exponent is written over a quantity, the exponent 1 is always understood. Thus, a^1 and a signify the same thing.

Exponents may be attached to figures as well as letters. Thus the product of 3 by 3 may be written 3^2, which equals 9.

"	$3\times3\times3$	"	3^3,	"	27.
"	$3\times3\times3\times3\times3$	"	3^5,	"	243.

17. The *sign of evolution*, or the *radical sign*, is the character $\sqrt{\ }$. When placed over a quantity, it indicates that a root of that quantity is to be extracted. The name or *index* of the required root is the number written above the radical sign. Thus,

$\sqrt[2]{9}$, or simply $\sqrt{9}$, denotes the square root of 9, which is 3.

$\sqrt[3]{64}$ denotes the cube root of 64, which is 4.

$\sqrt[4]{16}$ denotes the fourth root of 16, which is 2.

So, also,

$\sqrt[2]{a}$, or simply $\sqrt{a}$, denotes the square root of a.

$\sqrt[4]{a}$ denotes the fourth root of a.

$\sqrt[n]{a}$ denotes the nth root of a, where n may represent any number whatever.

When no index is written over the sign, the index 2 is understood. Thus, instead of $\sqrt[2]{ab}$, we usually write $\sqrt{ab}$.

Symbols which indicate Relation.

18. The *sign of equality* consists of two short horizontal lines, $=$. When written between two quantities, it indicates that they are equal to each other.

Thus, $7+6=13$ denotes that the sum of 7 and 6 is equal to 13.

In like manner, $a=b+c$ denotes that a is equal to the sum of b and c; and $a+b=c-d$ denotes that the sum of the numbers designated by a and b, is equal to the difference of the numbers designated by c and d.

19. The *sign of inequality* is the angle $>$ or $<$. When placed between two quantities, it indicates that they are unequal, the opening of the angle being turned toward the greater number. When the opening is toward the left, it is read *greater than;* when the opening is toward the right, it is read *less than.* Thus, $5>3$ denotes that 5 is greater than 3; and $6<11$ denotes that 6 is less than 11. So, also, $a>b$ denotes that a is greater than b; and $x<y+z$ denotes that x is less than the sum of y and z.

20. A *parenthesis*, ( ), or a *vinculum*, ——, is employed to connect several quantities, all of which are to be subjected to the *same operation.*

Thus the expression $(a+b+c)\times x$, or $\overline{a+b+c}\times x$, indicates that the sum of a, b, and c is to be multiplied by x. But $a+b+c\times x$ denotes that c *only* is to be multiplied by x.

When the parenthesis is used, the sign of multiplication is generally omitted. Thus, $(a+b+c)\times x$ is the same as $(a+b+c)x$.

21. The *sign of ratio* consists of two points like the colon : placed between the quantities compared. Thus the ratio of a to b is written $a:b$.

22. The *sign of proportion* consists of a combination of the sign of ratio and the sign of equality, thus, : = :; or a combination of eight points, thus, : :: :.

Thus, if a, b, c, d, are four quantities which are proportional to each other, we say a is to b as c is to d; and this is expressed by writing them thus:

$$a:b=c:d,$$

or

$$a:b::c:d.$$

23. The *sign of variation* is the character $\propto$. When written between two quantities, it denotes that both increase or diminish together, and in the same ratio. Thus the expression $s\propto tv$ denotes that s varies in the same ratio as the product of t and v.

24. Three dots $\therefore$ are sometimes employed to denote *therefore*, or *consequently*.

A few other symbols are employed in Algebra, in addition to those here enumerated, which will be explained as they occur.

Combination of Algebraic Quantities.

25. Every number written in algebraic language—that is, by aid of algebraic symbols—is called an *algebraic quantity*, or an *algebraic expression*.

Thus, $3a^2$ is the algebraic expression for three times the square of the number a.

$7a^3b^4$ is the algebraic expression for seven times the third power of a, multiplied by the fourth power of b.

26. An algebraic quantity, not composed of parts which are separated from each other by the sign of addition or subtraction, is called a *monomial*, or a quantity of *one term*, or simply *a term*.

Thus, $3a$, $5bc$, and $7xy^2$, are monomials.

Positive terms are those which are preceded by the sign plus, and *negative terms* are those which are preceded by the sign minus. When the first term of an algebraic quantity is positive, the sign is generally omitted. Thus $a+b-c$ is the same as $+a+b+c$. The sign of a negative term should never be omitted.

27. The *coefficient* of a quantity is the number or letter prefixed to it, showing how often the quantity is to be taken.

Thus, instead of writing $a+a+a+a+a$, which represents 5 a's added together, we write $5a$, where 5 is the coefficient of a. In $6(x+y)$, 6 is the coefficient of $x+y$. When no coefficient is expressed, 1 is always to be understood. Thus, $1a$ and a denote the same thing.

The coefficient may be a *letter* as well as a figure. In the expression nx, n may be considered as the coefficient of x, because x is to be taken as many times as there are units in n. If n stands for 5, then nx is 5 times x. When the coefficient is a number, it may be called a *numerical* coefficient; and when it is a letter, a *literal* coefficient.

In $7ax$, 7 may be regarded as the coefficient of ax, or $7a$ may be regarded as the coefficient of x.

28. The coefficient of a positive term shows how many times the quantity is taken *positively*, and the coefficient of a negative term shows how many times the quantity is taken *negatively*. Thus,

$$+4x=+x+x+x+x;$$
$$-4x=-x-x-x-x.$$

29. *Similar terms* are terms composed of the same letters, affected with the same exponents. The signs and coefficients may differ, and the terms still be similar.

Thus, $3ab$ and $7ab$ are similar terms.

Also, $5a^2c$ and $-3a^2c$ are similar terms.

30. *Dissimilar terms* are those which have different letters or exponents.

Thus, axy and axz are dissimilar terms.

Also, $3ab^2$ and $4a^2b$ are dissimilar terms.

31. A *polynomial* is an algebraic expression consisting of more than one term; as, $a+b$; or $a+2b-5c+x$.

A polynomial consisting of two terms only is usually called a *binomial;* and one consisting of three terms only is called a *trinomial.* Thus, $3a+5b$ is a binomial; and $5a-3bc+xy$ is a trinomial.

32. The *degree* of a term is the number of its literal factors.

Thus, $3a$ is a term of the first degree.

$5ab$	"	second "
$6a^2bc^3$	"	sixth "

In general, the degree of a term is found by taking the sum of the exponents of all the letters contained in the term.

Thus the degree of the term $5ab^2cd^3$ is $1+2+1+3$, or 7; that is, this term is of the seventh degree.

33. A polynomial is said to be *homogeneous* when all its terms are of the same degree.

Thus, $3a^2-4ab+b^2$ is of the second degree, and homogeneous.
$2a^3+3a^2c-4c^2d$ " third " "
But $5a^3-2ab+c$ is *not* homogeneous.

34. The *reciprocal* of a quantity is the quotient arising from dividing a unit by that quantity.

Thus the reciprocal of 2 is $\frac{1}{2}$; the reciprocal of a is $\frac{1}{a}$.

35. A *function* of a quantity is any expression containing that quantity. Thus,

ax^2+b is a function of x.
ay^3+cy+d is a function of y.
ax^2-by^2 is a function of x and y.

Exercises in Algebraic Notation.

36. In the following examples the pupil is simply required to express given relations in algebraic language.

Ex. 1. Give the algebraic expression for the following statement: The second power of a, increased by twice the product of a and b, diminished by c, and increased by d, is equal to fifteen times x. *Ans.* $a^2+2ab-c+d=15x$.

Ex. 2. The quotient of three divided by the sum of x and four, is equal to twice b diminished by eight.

Ex. 3. One third of the difference between six times x and four, is equal to the quotient of five divided by the sum of a and b.

Ex. 4. Three quarters of x increased by five, is equal to three sevenths of b diminished by seventeen.

Ex. 5. One ninth of the sum of six times x and five, added to one third of the sum of twice x and four, is equal to the product of a, b, and c.

Ex. 6. The quotient arising from dividing the sum of a and b by the product of c and d, is greater than four times the sum of m, n, x, and y.

37. In the following examples the pupil is required to translate the algebraic symbols into common language.

Ex. 1. $\frac{a+x}{b}+\frac{x}{c}=\frac{m}{a+b}$.

Ans. The quotient arising from dividing the sum of a and x by b, increased by the quotient of x divided by c, is equal to the quotient of m divided by the sum of a and b.

Ex. 2. $7a^2+(b-c)\times(d+e)=x+y$.

How should the preceding example be read when the first parenthesis is omitted?

Ex. 3. $\frac{a+x}{3+b-c}+\frac{6-4y}{3}=\frac{m}{a+7}$.

Ex. 4. $4\sqrt{ab}-25=\frac{2b+c}{3a-d}$.

Ex. 5. $2a\sqrt{b^3}-ac=5(a+m+x)$.

Ex. 6. $\frac{\sqrt{5b}+3\sqrt{c}}{1+2a}=5x+\frac{1}{4}$.

Computation of Numerical Values.

38. The *numerical value* of an algebraic expression is the result obtained when we assign particular values to all the letters, and perform the operations indicated.

Suppose the expression is $2a^2b$.

If we make $a=2$ and $b=3$, the value of this expression will be $2\times2\times2\times3=24$.

If we make $a=4$ and $b=3$, the value of the same expression will be $2\times4\times4\times3=96$.

The numerical value of a polynomial is not affected by *changing the order* of the terms, provided we preserve their respective signs.

The expressions $a^2+2ab+b^2$, a^2+b^2+2ab, and $b^2+2ab+a^2$, have all the same numerical value.

Find the numerical values of the following expressions, in which $a=6$, $b=5$, $c=4$, $m=8$, and $n=2$.

Ex. 1. $a^2+3ab-c^2$. *Ans.* $36+90-16=110$.

Ex. 2. $a^2\times(a+b)-2abc$. *Ans.* 156.

Ex. 3. $\dfrac{a^3}{a+3c}+c^2.$ *Ans.* 28.

Ex. 4. $c+\dfrac{2bc}{\sqrt{2ac+c^2}}.$

Ex. 5. $\sqrt{b^2-ac}+\sqrt{2ac+c^2}.$

Ex. 6. $3\sqrt{c}+2a\sqrt{2a+b+2c}.$

Ex. 7. $(3\sqrt{c}+2a)\sqrt{2a+b+2c}.$

Ex. 8. $\dfrac{36}{a}-\dfrac{15}{b}+\dfrac{12}{m-n}-\dfrac{30}{m+n}.$

Ex. 9. $\dfrac{a^2b^2c^2m^2+3abcm+2}{abcm+1}.$

Find the numerical values of the following expressions, in which $a=3$, $b=5$, $c=2$, $m=4$, $n=6$, and $x=9$.

Ex. 10. $\dfrac{ax^2+b^2+m}{bx-a^2-c}.$ *Ans.* 8.

Ex. 11. $5x-7\sqrt{x}.$ *Ans.* 24.

Ex. 12. $2x^2+\sqrt{2x^2+7}.$ *Ans.* 175.

Ex. 13. $\sqrt{10+n}-\sqrt[4]{10+n}.$

Ex. 14. $5(m^2+n^2)+4ax.$

Ex. 15. $a^4-4a^3+7a^2-6a.$

Ex. 16. $\sqrt{5\sqrt{m}+5\sqrt{x}}+\sqrt{m}+\sqrt{x}.$

Ex. 17. $\dfrac{m^2+n^2-b^2}{m-n+b}+\dfrac{acmn}{m+2n}-\dfrac{4x^2-7ab+1}{2a+m}.$

Ex. 18. $m\left(\dfrac{a}{c}+\dfrac{b}{m}+\dfrac{2x}{n}\right).$

Ex. 19. $\dfrac{n^3-m^3}{m^2+mn+n^2}.$

CHAPTER II.

ADDITION.

39. *Addition*, in Algebra, is the connecting of quantities together by means of their proper signs, and incorporating such as can be united into one *sum*.

When the Quantities are similar and have the same Signs.

40. The sum of $3a$, $4a$, and $5a$, is obviously $12a$. That is,

$$3a+4a+5a=12a.$$

So, also, $-3a$, $-4a$, and $-5a$, make $-12a$; for the minus sign before each of the terms shows that they are to be subtracted, not from each other, but from some quantity which is not here expressed; and if $3a$, $4a$, and $5a$ are to be subtracted successively from the same quantity, it is the same as subtracting at once $12a$. Hence we deduce the following

RULE.

Add the coefficients of the several quantities together, and to their sum annex the common letter or letters, prefixing the common sign.

EXAMPLES.

$3a$	$-3ab$	$2b+3x$	$a-2x^2$	$2a+y^2$
$5a$	$-6ab$	$5b+7x$	$4a-3x^2$	$5a+2y^2$
$7a$	$-ab$	$b+2x$	$3a-5x^2$	$9a+3y^2$
a	$-7ab$	$4b+3x$	$7a-x^2$	$4a+6y^2$
$16a$	$-17ab$			

The pupil must continually bear in mind the remark of *Art.* 26, that, when no sign is prefixed to a quantity, plus is always to be understood.

When the Quantities are similar, but have different Signs.

41. The expression $7a-4a$ denotes that $4a$ is to be subtracted from $7a$, and the result is obviously $3a$. That is,

$$7a-4a=3a.$$

The expression $5a-2a+3a-a$ denotes that we are to subtract $2a$ from $5a$, add $3a$ to the remainder, and then subtract a from the last sum, the result of which operation is $5a$. That is,

$$5a-2a+3a-a=5a.$$

It is generally most convenient to take the sum of the positive quantities, which in the preceding case is $8a$; then take the sum of the negative quantities, which in this case is $3a$; and we have $8a-3a$, or $5a$, the same result as before. Hence we deduce the following

RULE.

Add all the positive coefficients together, and also all those that are negative; subtract the least of these results from the greater; to the difference annex the common letter or letters, and prefix the sign of the greater sum.

EXAMPLES.

$-3a$	$6x+5ay$	$2ay-7$	$-2a^2x$	$-6a^2+2b$
$+7a$	$-3x+2ay$	$-ay+8$	a^2x	$2a^2-3b$
$+8a$	$x-6ay$	$2ay-9$	$-3a^2x$	$-5a^2-8b$
$-a$	$2x+ay$	$3ay-11$	$7a^2x$	$4a^2-2b$
$+11a$	$6x+2ay$			

When some of the Quantities are dissimilar.

42. Dissimilar terms can not be united into one term by addition.

Thus $2a$ and $3b$ neither make $5a$ nor $5b$. Their sum can therefore only be *indicated* by connecting them by their proper signs, thus, $2a+3b$.

In adding together polynomials which contain several groups of similar quantities, it is most convenient to write them in such a manner that each group of similar quantities may occupy a column by itself. Hence we deduce the following

RULE.

Write the quantities to be added so that the similar terms may be arranged in the same column.

Add up each column separately, and connect the several results by their proper signs.

EXAMPLES.

1. Add $2a+3b+4c$, $a+2b+5c$, $3a-b+2c$, and $-a+4b-6c$.
Ans. $5a+8b+5c$.

2. Add $2xy-2x^2+y^2$, $3x^2+xy+4y^2$, $x^2-xy+3y^2$, and $4x^2-2y^2-3xy$. *Ans.* $6x^2-xy+6y^2$.

3. Add $5a^2x^2-2xy$, $3ax-4xy$, $7xy-4ax$, a^2x^2+5xy, and $2ax-3xy$. *Ans.* $6a^2x^2+3xy+ax$.

4. Add $2a^2-3ac+3b-cd$, $4a^2-ac+2cd-b$, $3a^2+2ac-4b+3cd$, and $a^2-2ac+5cd-2b$. *Ans.* $10a^2-4ac-4b+9cd$.

5. Add $7m+3n-14x$, $3a+9n-11m$, $5x-4m+8n$, and $11n-2b-m$. *Ans.* $3a-2b-9m+31n-9x$.

6. Add $2a^2x+3ax^2+x^2$, $2ax-3a^2-4x^2$, $-2x^2+3a^2x-5ax^2$, and $3a^2-2a^2x+2ax^2$. *Ans.* $3a^2x-5x^2+2ax$.

7. Add $2a^2b^2-3ax+5m^2y$, $2m^2y+3a^2b^2-2ax$, $4ax-3m^2y-4a^2b^2$, and $ax+3a^2b^2-4m^2y$. *Ans.* $4a^2b^2$.

8. Add $7ab^3-12ax^2$, $13ab^3+ax$, $3ax^2-8ab^3$, and $-12ab^3+9ax^2-5$.

9. Add $4x^2+2ax+1$, $3ax-2x^2+5$, $3x^2-6ax+4$, and $5x^2+ax-1$.

10. Add $2a^3x^2-3a^2x^3-a^2x-ax^2+2ax$, $4a^2x^3-3a^2x+4a^3x^2+2ax^2+3ax$, and $3a^3x^2-ax^2-a^2x^3+4a^2x+ax$.

11. Add $14a^3x-7a^2b^2+3a^2$, $5a^2b^2c^2+3a^2b^2+2a^2$, $2a^2b^2c^2-5a^3x-a^2$, and $4a^2b^2-9a^3x-4a^2$.

12. Add $ax^4-bx^3+cx^2-7$, $2bx^3+3cx^2-4x+1$, $3ax^4-4bx^3-2cx^2+3$, and $2ax^4+3bx^3-2cx^2+3$.

43. It must be observed that the term addition is used in a *more general sense* in Algebra than in Arithmetic. In Arithmetic, where all quantities are regarded as positive, addition implies *augmentation.* The sum of two quantities will therefore be numerically greater than either quantity. Thus the sum of 7 and 5 is 12, which is numerically greater than either 5 or 7.

But in Algebra, the quantities to be added may be either positive or negative; and by the sum of two quantities we understand their aggregate, taken with reference to their signs.

Thus the sum of $+7$ and -5 is $+2$, which is numerically less than either 7 or 5. So, also, the sum of $+a$ and $-b$ is $a-b$. In this case the *algebraic sum* is numerically the *difference* of the two quantities.

This is one instance among many in which the same terms are used in a much more general sense in the higher mathematics than they are in Arithmetic.

44. When dissimilar terms have a common literal part, we may regard the other factors as the *coefficient* of the common letter or letters. The sum of the terms will then be expressed by inclosing the sum of the coefficients in a parenthesis, and prefixing it to the common letter or letters.

Thus the sum of ax^2, bx^2, and cx^2 may be written

$$(a+b+c)x^2.$$

EXAMPLES.

1. Add ax, $2bx$, and $3mx$. *Ans.* $(a+2b+3m)x$.
2. Add $3axy^2$, $2bxy^2$, and $-5axy^2$. *Ans.* $(2b-2a)xy^2$.
3. Add $2ax+3y$, $5ax-y$, and $x-4y$. *Ans.* $(7a+1)x-2y$.
4. Add $2x+3xy$, $ax+bxy$, and $bx+3mxy$.
5. Add $mx+ny$, $3ax-2y$, and $4bx+ay$.
6. Add $4m\sqrt{x}+3$, $2a\sqrt{x}-1$, and $b\sqrt{x}+y$.
7. Add $3ax^2+2bx-1$, $4bx^2-ax+3$, and mx^2-nx+5.
8. Add $2ax^4+3bx^3-7$, $3mx^4-nx^3+2$, and $4x^4-ax^3+1$.
9. Add amx^3+bnx^2+cx, bmx^3-anx^2+ax, and cmx^3-nx^2+3bx.
10. Add $(a-b)\sqrt{x}$ and $(a+b-c)\sqrt{x}$.

CHAPTER III.

SUBTRACTION.

45. *Subtraction* is the operation of finding the difference between two quantities or sets of quantities. The quantity to be subtracted is called the *subtrahend;* the quantity from which it is to be subtracted is called the *minuend;* the quantity which is left after the subtraction is called the *remainder.*

Let it be required to subtract $8-3$ from 15.

Now $8-3$ is equal to 5; and 5 subtracted from 15 leaves 10. The result, then, must be 10. But, to perform the operation on the numbers as they were given, we first subtract 8 from 15, and obtain 7. This result is *too small* by 3, because the number 8 is *larger* by 3 than the number which was required to be subtracted. Therefore, in order to correct this result, the 3 must be added, and the operation may be expressed thus,

$$15-8+3=10.$$

Again, let it be required to subtract $c-d$ from $a-b$. It is plain that, if the part c were alone to be subtracted, the remainder would be

$$a-b-c.$$

But, since the quantity actually proposed to be subtracted is *less* than c by d, *too much* has been taken away by d, and therefore the true remainder will be *greater* than $a-b-c$ by d, and may hence be expressed thus,

$$a-b-c+d,$$

where the signs of the last two terms are both *contrary* to what they were given in the subtrahend. Hence we perceive that a quantity is subtracted by simply changing its *sign.* In practice it is most convenient to write the quantities so that similar terms may be found in the same column.

46. Hence we deduce the following

RULE.

Write the subtrahend under the minuend, arranging similar terms in the same column.

Conceive the signs of all the terms of the subtrahend to be changed from + to —, or from — to +, and then proceed as in addition.

EXAMPLES.

	1.	2.	3.	4.	5.
From	$15ax^2$	$5abx^2$	$4abx^3$	$-3abx^2$	$-12a^2bc$
Subtract	$7ax^2$	$11abx^2$	$-3abx^3$	$-8abx^2$	$3a^2bc$
Remainder	$8ax^2$	$-6abx^2$	$7abx^3$	$5abx^2$	$-15a^2bc$

	6.	7.	8.
From	$3a^2+4b-2x$	$5x^2-3ax^2+11$	$a^2+2ab+b^2$
Subtract	$a^2+7b-8x$	$7x^2-8ax^2-\ 2$	$a^2-2ab+b^2$
Remainder	$2a^2-3b+6x$	$-2x^2+5ax^2+13$	$4ab$

9. From $3a+5b-2c$ subtract $2a-b$. *Ans.* $a+6b-2c$.
10. From $5abc-2b-6$ subtract $3abc-2b+1$. *Ans.* $2abc-7$.
11. From $4a^2-7a+3x$ subtract $a^2+3a-2x$. *Ans.* $3a^2-10a+5x$.
12. From $a-b+2m-x$ subtract $3x+m-4b+a$.
13. From $2x^3-x^2y+5xy^2$ subtract $x^3-2xy^2+y^3$.
14. From $m+n$ subtract $m-n$.
15. From $m+n+x$ subtract $-m-n-x$.
16. From $5a^2-3a-7$ subtract $-2a^2-4a+10$.
17. From $m^4+3m^3-4m^2-2m+1$ subtract $m^4-2m^3+m^2-3m+5$.
18. From $x^5-5x^4+10x^3-3$ subtract $x^5+5x^4-10x^3+3$.
19. From $3a^2+ax+2x^2-14a^2x$ subtract $x^2-15a^2x+2a^2-4ax$.
20. From $6abx-4mn+5ax$ subtract $3mn+6ax-3abx$.

Subtraction may be *proved*, as in Arithmetic, by adding the remainder to the subtrahend. The sum should be equal to the minuend.

47. It will be perceived that the term subtraction is used in a *more general sense* in Algebra than in Arithmetic. In Arithmetic, where all quantities are regarded as positive, a number is always *diminished* by subtraction. But in Algebra the difference between two quantities may be numerically greater than either. Thus the difference between $+a$ and $-b$ is $a+b$.

The distinction between positive and negative quantities may be illustrated by the scale of a thermometer. The degrees above zero are considered positive, and those below zero negative. From five degrees above zero to five degrees below zero, the numbers stand thus,

$$+5,\ +4,\ +3,\ +2,\ +1,\ 0,\ -1,\ -2,\ -3,\ -4,\ -5.$$

The difference between a temperature five degrees *above* zero and one which is five degrees *below* zero, is ten degrees, which is numerically the *sum* of the two quantities. Ten is said to be the *algebraic difference* between $+5$ and -5.

48. When dissimilar terms have a *common literal part*, the difference of the terms may be expressed, as in *Art.* 44, by inclosing the difference of the coefficients in a parenthesis, and prefixing it to the common letter or letters.

Thus the difference between ax^2 and bx^2 may be written

$$(a-b)x^2.$$

EXAMPLES.

1. From ax^2y^2 subtract $-3x^2y^2$. *Ans.* $(a+3)x^2y^2$.
2. From $2ax+3y$ subtract $5bx-y$. *Ans.* $(2a-5b)x+4y$.
3. From $mx+ny$ subtract $3ax-2y$. *Ans.* $(m-3a)x+(n+2)y$.
4. From $4m\sqrt{x}+3$ subtract $2a\sqrt{x}-1$.
5. From $2ax^4+3bx^3-7$ subtract $3mx^4-nx^3+2$.
6. From amx^3+bnx^2+cx subtract bmx^3-anx^2+ax.
7. From $m+am+bm$ subtract $am+bm+cm$.
8. From $1+3ax^2+5a^2x^3+7a^3x^4$ subtract $x^2-3ax^3-5a^2x^4$.

49. *Use of the Parenthesis.*—If we wish to *indicate* that one polynomial is to be subtracted from another, we may inclose it in a parenthesis, and prefix the sign minus. Thus the expression

$$a-b-(m-n+x)$$

indicates that the polynomial $m-n+x$ is to be *subtracted* from the polynomial $a-b$. Performing the operation indicated, we have

$$a-b-m+n-x.$$

The expression $a-b+(m-n+x)$

indicates that the polynomial $m-n+x$ is to be *added* to the polynomial $a-b$, and the result is

$$a-b+m-n+x.$$

Hence we see that a parenthesis preceded by the *plus* sign may be removed without changing the signs of the inclosed terms; and, conversely, any number of terms, with their proper signs, may be inclosed in a parenthesis, and the plus sign written before the whole.

But if the parenthesis is preceded by the *minus* sign, the signs of all the inclosed terms must be *changed* when the parenthesis is removed; and, conversely, any number of terms may be inclosed in a parenthesis, and preceded by the minus sign, provided the signs of all the inclosed terms are changed.

50. According to the preceding principle, polynomials may be written in a variety of forms.

Thus, $a-b-c+d$
is equivalent to $a-(b+c-d)$,
or to $a-b-(c-d)$,
or to $a+d-(b+c)$.
These expressions are all equivalent, the first form being the simplest.

EXAMPLES.

Reduce the following expressions to their simplest forms.

1. $2a^3-5a^2b+3ab^2-(a^3+b^3-ab^2)$.
Ans. $a^3-5a^2b+4ab^2-b^3$.

2. $a+b+c-(a-b)-(b-c)$. *Ans.* $b+2c$.
3. $4a^2-b-(2a-3b+1)+3a$. *Ans.* $4a^2+a+2b-1$.
4. $a+2b-(3m-2a+3x^2)$.
5. $3a^3-2a^2+a+1-(2a^3-a^2-a+5)-(a^3-a^2-5a-4)$.
6. $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$.
7. $8a^2xy-5bx^2y+7cxy^2-3y^5-(a^2xy+3bx^2y-4cxy^2+9y^5)$.
8. $7ax^2-10a^2x^2+11a^2x-5a^4-(9ax^2+12a^2x^2-6a^2x-9a^4)$.

51. Hence we see that when an expression is inclosed in a parenthesis, the *essential sign* of a term depends not merely upon the sign which immediately precedes it, but also upon the sign preceding the parenthesis.

Thus	$m+(+n)$	is equivalent to	$m+n$,
and	$m+(-n)$	"	$m-n$.
But	$m-(+n)$	"	$m-n$,
and	$m-(-n)$	"	$m+n$.

The sign immediately preceding n is called the *sign of the quantity;* the sign preceding the parenthesis may be called the *sign of the operation;* while the sign resulting from the operation is called the *essential sign* of the term. We perceive that when the sign of the quantity is the *same* as the sign of operation, the essential sign of the term is *positive;* but when the sign of the quantity is *different* from the sign of operation, the essential sign of the term is *negative.*

52. *Use of Negative Quantities.*—The introduction of negative quantities into Algebra enables us not only to compare the *magnitude*, but also to indicate the *relation* or *quality* of the objects about which we are reasoning. This peculiarity will be understood from a few examples:

1*st. Gain and Loss in Trade.*—Suppose a merchant to gain in one year a certain sum, and in the following year to lose a certain sum; we are required to determine what change has taken place in his capital. This may be indicated algebraically by regarding the gains as positive quantities, and the losses as negative quantities. Thus, suppose a merchant, with a capital of 10,000 dollars, loses 3000 dollars, afterward gains 1000 dollars, and then loses again 4000 dollars, the whole may be expressed algebraically thus,

$$10{,}000-3000+1000-4000,$$

which reduces to $+4000$. The $+$ sign of the result indicates that he has now 4000 dollars remaining in his possession. Suppose he further gains 500 dollars, and then loses 7000 dollars. The whole may now be expressed thus,

$$10{,}000-3000+1000-4000+500-7000,$$

which reduces to -2500. The $-$ sign of the result indicates that his losses exceed the sum of all his gains and the property originally in his possession; that is, he owes 2500 dollars more than he can pay.

53. *2d. Motion in Contrary Directions.*—Suppose a ship to sail alternately northward and southward, and we are required to determine the last position of the ship. This may be indicated algebraically, if we agree to consider motion in *one direction* as a *positive* quantity, and motion in the *opposite direction* as a *negative* quantity.

Suppose a ship, setting out from the equator, sails northward 50 miles, then southward 30 miles, then northward 10 miles, then southward again 20 miles, and we wish to determine the last position of the ship. If we call the northerly motion $+$, the whole may be expressed algebraically thus,

$$50-30+10-20,$$

which reduces to $+10$. The positive sign of the result indicates that the ship was *north* of the equator by 10 miles.

Suppose the same ship sails again 40 miles north, then 70 miles south, the whole may be expressed thus,

$$50-30+10-20+40-70,$$

which reduces to -20. The negative sign of the result indicates that the ship was now *south* of the equator by 20 miles.

We have here regarded the northerly motion as $+$, and the southerly motion as $-$; but we might with equal propriety have regarded the northerly motion as $-$, and the southerly motion as $+$. It is, however, indispensable that we adhere to the same system throughout, and retain the proper sign of the result, since this sign shows whether the ship was at any time north or south of the equator.

In the same manner, if we regard westerly motion as $+$, we must regard easterly motion as $-$, and *vice versa;* and, generally, when quantities which are estimated in different directions enter into the same algebraic expression, those which are measured in one direction being treated as $+$, those which are measured in the opposite direction must be regarded as $-$.

54. The same principle is applicable to a great variety of examples in Geography, Astronomy, etc. Thus, north latitude is generally indicated by the sign $+$, and south latitude by the sign $-$. West longitude is indicated by the sign $+$, and east longitude by the sign $-$.

Degrees of a thermometer above zero are indicated by the sign $+$, while degrees below zero are indicated by the sign $-$.

A variation of the magnetic needle to the west is indicated by the sign $+$, while a variation to the east is indicated by the sign $-$.

The date of an event since the birth of Christ is indicated by the sign $+$; the date of an event before the birth of Christ, by the sign $-$; and the same distinction is observed in a great variety of cases which occur in the application of the mathematics to practical problems. In all such cases the positive and negative signs enable us not merely to compare the *magnitude*, but also to indicate the *relation* of the quantities considered.

CHAPTER IV.

MULTIPLICATION.

55. *Multiplication* is the operation of repeating one quantity as many times as there are units in another.

The quantity to be multiplied is called the *multiplicand;* and that by which it is to be multiplied is called the *multiplier.*

56. When several quantities are to be multiplied together, the result will be the same in whatever *order* the multiplication is performed.

In order to demonstrate this principle, let unity be repeated five times upon a horizontal line, and let there be formed four such parallel lines, thus,

• • • • •
• • • • •
• • • • •
• • • • •

Then it is plain that the number of units in the table is equal to the five units of the horizontal line repeated as many times as there are units in a vertical column; that is, to the product of 5 by 4. But this sum is also equal to the four units of a vertical line repeated as many times as there are units in a horizontal line; that is, to the product of 4 by 5. Therefore the product of 5 by 4 is equal to the product of 4 by 5. For the same reason, $2\times3\times4$ is equal to $2\times4\times3$, or $4\times3\times2$, or $3\times4\times2$, the product in each case being 24. So, also, if a, b, and c represent any three numbers, we shall have abc equal to bca or cab.

CASE I.

When both the factors are monomials.

57. Suppose it is required to multiply $5a$ by $4b$. The product may be indicated thus, $5a\times4b$.

But since the order of the factors may be changed without affecting the value of the product, the factors of the same kind may be written together thus,

$$4 \times 5ab;$$

or, simplifying the expression, we have

$$20ab.$$

Hence we see that *the coefficient of the product is equal to the product of the coefficients of the multiplicand and multiplier.*

58. *The Law of Exponents.*—We have seen, in *Art.* 16, that when the same letter appears several times as a factor in a product, this is briefly expressed by means of an exponent. Thus, aaa is written a^3, the number 3 showing that a enters three times as a factor. Hence, if the same letters are found in two monomials which are to be multiplied together, the expression for the product may be abbreviated by adding the exponents of the same letter. Thus, if we are to multiply a^3 by a^2, we find a^3 equivalent to aaa, and a^2 to aa. Therefore the product will be $aaaaa$, which may be written a^5, a result which is obtained by adding together 3 and 2, the exponents of the common letter a. Hence we see that *the exponent of any letter in the product is equal to the sum of the exponents of this letter in the multiplicand and multiplier.*

59. Hence, for the multiplication of monomials, we have the following

RULE.

Multiply together the coefficients of the two terms for the coefficient of the product.

Write after this all the letters in the two monomials, giving to each letter an exponent equal to the sum of its exponents in the two factors.

EXAMPLES.

	1.	2.	3.	4.
Multiply	$7abc$	$5a^2b^3c$	$9amx^4y$	$6a^2b^3c^4$
by	$5mn$	$3ab^2c$	$4am^2xy^3$	$8a^3bc^2$
Product	$35abcmn$	$15a^3b^5c^2$	$36a^2m^3x^5y^4$	$48a^5b^4c^6$

5. Multiply $9a^3x$ by $7a^3y$.
6. Multiply $12a^3b^4c^5$ by $11a^5b^4c^3$.
7. Multiply a^m by a^n. *Ans.* a^{m+n}.
8. Multiply $8a^m$ by $9a^n$.
9. Multiply a^m by a^m. *Ans.* a^{2m}.
10. Multiply $9a^m$ by $12a^m$.
11. Multiply a^mb by ab^m. *Ans.* $a^{m+1}b^{m+1}$.
12. Multiply $6a^mx^n$ by $5a^mx$. *Ans.* $30a^{2m}x^{n+1}$.
13. Multiply $3a^{2m}x^n$ by $4a^mx^{3n}$. *Ans.* $12a^{3m}x^{4n}$.
14. What is the continued product of $5a$, $4m^2x$, and $9a^2m^3x$? *Ans.* $180a^3m^5x^2$.
15. What is the continued product of $7a^2b$, ab^5, and $4ac^3$?
16. What is the continued product of $3a^mx$, $5ab^2$, and $7abx$?
17. What is the continued product of a, ab, abc, $abcd$, and $abcdx$?
18. What is the continued product of a^3, a^3b^2, a^3bc, and $a^3b^5c^4x$?

CASE II.

When one or both of the factors are polynomials.

60. Represent the sum of the positive terms of any polynomial by a, and the sum of the negative terms by b. Then $a-b$ will represent any polynomial whatever. In like manner, $c-d$ will represent any other polynomial whatever. It is required to find the product of $a-b$ by $c-d$.

In the first place, let us multiply $a-b$ by c. This implies that the *difference* of the units in a and b is to be repeated c times. If $+a$ be repeated as many times as there are units in c, the result will be $+ac$. Also, if $-b$ be repeated as many times as there are units in c, the result will be $-bc$, for $-b$ taken twice is $-2b$, taken three times is $-3b$, etc.; and if it be repeated c times, the result will be $-cb$ or $-bc$. The entire operation may be exhibited thus:

$$\begin{array}{r} a-b \\ c \\ \hline ac-bc. \end{array}$$

Next let us multiply $a-b$ by $c-d$. When we multiply

$a-b$ by c, we obtain $ac-bc$. But $a-b$ was only to be taken $c-d$ times; therefore, in this first operation, we have repeated it d times *more* than was required. Hence, to have the true product, we must *subtract* d times $a-b$ from $ac-bc$. But d times $a-b$ is equal to $ad-bd$, which, subtracted from $ac-bc$, gives

$$ac-bc-ad+bd.$$

If the pupil does not perceive the force of this reasoning, it will be best to repeat the argument with numbers, thus: Let it be proposed to multiply $8-5$ by $6-2$; that is, the quantity $8-5$ is to be repeated as many times as there are units in $6-2$. If we multiply $8-5$ by 6, we obtain $48-30$; that is, we have repeated $8-5$ six times. But it was only required to repeat the multiplicand *four* times, or $6-2$. We must therefore *diminish* this product by twice $8-5$, which is $16-10$; and this subtraction is performed by changing the signs of the subtrahend. Hence we have

$$48-30-16+10,$$

which is equal to 12. This result is obviously correct; for $8-5$ is equal to 3, and $6-2$ is equal to 4; that is, it was required to multiply 3 by 4, the result of which is 12, as found above.

We have thus obtained the following results:

$$+a\times(+b)=+ab,$$
$$+a\times(-b)=-ab,$$
$$-a\times(+b)=-ab,$$
$$-a\times(-b)=+ab,$$

from which we perceive that *when the two factors have like signs, the product is positive; and when the two factors have unlike signs, the product is negative.*

61. Hence, for the multiplication of polynomials, we have the following general

RULE.

Multiply each term of the multiplicand by each term of the mul-

tiplier, and add together all the partial products, observing that like signs require + in the product, and unlike signs −.

EXAMPLES.

	1.	2.
Multiply	$2a+3b$	$a^2+2ab+b^2$
by	$4a-5b$	$a+b$
Partial products	$8a^2+12ab$	$a^3+2a^2b+ab^2$
	$-10ab-15b^2$	$a^2b+2ab^2+b^3$
Result	$8a^2+2ab-15b^2$	$a^3+3a^2b+3ab^2+b^3$

It is immaterial in what order the terms of a polynomial are arranged, or in what order the letters of a term are arranged. It is, however, generally most convenient to arrange the letters of a term *alphabetically*, and to arrange the terms of a polynomial in the order of the powers of some common letter.

3. Multiply a^2-ab+b^2 by $a+b$. *Ans.* a^3+b^3.
4. Multiply $a^2-2ab+b^2$ by $a-b$.
5. Multiply $3a^2-2a+5$ by $a-4$.
6. Multiply a^2-ab+b^2 by a^2+ab+b^2. *Ans.* $a^4+a^2b^2+b^4$.
7. Multiply $2a^2-3ab+4$ by $a^2+2ab-3$.
8. Multiply $a^3+a^2b+ab^2+b^3$ by $a-b$.
9. Multiply $a+mb$ by $a+nb$.
10. Multiply $3a+2bx-3x^2$ by $3a-2bx+3x^2$.
11. Multiply together $x-5$, $x+2$, and $x+3$.
12. Multiply together $x-3$, $x-4$, $x+5$, and $x-6$.
13. Multiply together a^2+ab+b^2, a^2-ab+b^2, and a^2-b^2.
14. Multiply together $a+x$, $b+x$, and $c+x$.
15. Multiply $a^4+a^3b+a^2b^2+ab^3+b^4$ by $a-b$.
16. Multiply $a^3-3a^2+7a-12$ by a^2+3a+2.
17. Multiply $x^4+2x^3+3x^2+2x+1$ by x^2-2x+1.
18. Multiply $14a^3x-6a^2bx+x^2$ by $14a^3x+6a^2bx-x^2$.
19. Multiply $x^3-x^2y+xy^2$ by $x^2-xy^2-y^3$.
20. Multiply $3x^2+8xy-5$ by $4x^2-7xy+9$.

62. *Degree of a Product.*—Since, in the multiplication of two monomials, every factor of both quantities appears in the product, it is obvious that the *degree* of the product will be equal to the *sum* of the degrees of the multiplier and multiplicand. Hence, also, if two polynomials are *homogeneous*, their product will be homogeneous.

Thus, in the first example of the preceding article, each term of the multiplicand is of the first degree, and also each term of the multiplier; hence each term of the product is of the second degree. For a similar reason, in the second example, each term of the product is of the third degree; and in the sixth example, each term of the product is of the fourth degree. This principle will assist us in guarding against errors in the multiplication of polynomials, so far as concerns the exponents.

63. *Number of Terms in a Product.*—When the product arising from the multiplication of two polynomials does not admit of any reduction of similar terms, the whole number of terms in the product is equal to the product of the numbers of the terms in the two polynomials. Thus, if we have five terms in the multiplicand, and four terms in the multiplier, the whole number of terms in the product will be 5×4, or 20. In general, if there be m terms in the multiplicand, and n terms in the multiplier, the whole number of terms in the product will be $m\times n$.

64. *Least Number of Terms in a Product.*—If the product of two polynomials contains *similar terms*, the number of terms in the product, when reduced, may be much less than mn; but it is important to observe that among the different terms of the product there are always two which *can not be combined* with any others. These are,

1st. The term arising from the multiplication of the two terms affected with the *highest* exponent of the same letter.

2d. The term arising from the multiplication of the two terms affected with the *lowest* exponent of the same letter.

For it is evident, from the rule of exponents, that these two partial products must involve the letter in question, the one with a *higher*, and the other with a *lower* exponent than any of the other partial products, and therefore can not be similar to any of them. Hence *the product of two polynomials can never contain less than two terms.*

65. For many purposes it is sufficient merely to *indicate* the multiplication of two polynomials, without actually performing the multiplication. This is effected by inclosing the polynomials in parentheses, and writing them in succession, with or without the sign $\times$. When the indicated multiplication has been actually performed, the expression is said to be *expanded.*

EXAMPLES.

1. Expand $(a+b)(c+d)$. *Ans.* $ac+bc+ad+bd$.
2. Expand $9a-7(b-c)$.
3. Expand and reduce
$$14(12-a-b-c)+13(4+a-c)-15(7-a-c).$$
4. Expand and reduce
$$28(a-b+c)+24(a+b-c)-13(b-a-c).$$
5. Expand and reduce
$$24a-6b-9(a+b)+25a-19(b-c)-17(a+b-c).$$
6. Expand and reduce
$$53(a-b+c)-27(a+b-c)-26(a-b-c).$$

66. The three following theorems have very important applications.

The square of the sum of two numbers is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.

Thus, if we multiply $a+b$
by $a+b$

$$\begin{array}{l} a^2+ab \\ \quad\;\; ab+b^2 \end{array}$$

we obtain the product $a^2+2ab+b^2$.

Hence, if we wish to obtain the square of a binomial, we can, according to this theorem, write out at once the terms of the result, without the necessity of performing an actual multiplication.

EXAMPLES.

1. $(3a+b)^2=$
2. $(3a+3b)^2=$
3. $(5a+3b)^2=$
4. $(5a^2+2b)^2=$
5. $(5a^3+b)^2=$
6. $(5a^2+7ab)^2=$
7. $(5a^3+8a^2b)^2=$
8. $(2a+\frac{1}{2})^2=$
9. $(1+\frac{1}{3})^2=$
10. $(3+\frac{1}{5})^2=$

67. *The square of the difference of two numbers is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Thus, if we multiply $a-b$
by $a-b$

$$\begin{array}{l} a^2-ab \\ \quad\;\; -ab+b^2 \\ \hline \end{array}$$

we obtain the product $a^2-2ab+b^2$.

EXAMPLES.

1. $(2a-3b)^2=$
2. $(5a-4b)^2=$
3. $(6a^2-x)^2=$
4. $(6a^2-3x)^2=$
5. $(x-\frac{1}{2}y)^2=$
6. $(7a^2-12ab)^2=$
7. $(7a^2b^2-12ab)^2=$
8. $(2a^3-5)^2=$
9. $(2-\frac{1}{3})^2=$
10. $(4-\frac{1}{5})^2=$

68. *Meaning of the sign* $\pm$.

Since $(a+b)^2=a^2+2ab+b^2$,

and $(a-b)^2=a^2-2ab+b^2$,

we may write both formulas in the following abbreviated form,

$$(a\pm b)^2=a^2\pm 2ab+b^2;$$

which indicates that, if we use the $+$ sign of b in the root, we must use the $+$ sign of $2ab$ in the square; but if we use the $-$ sign of b in the root, we must use the $-$ sign of $2ab$ in the square. By this notation we are enabled to express two distinct theorems by one formula.

69. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

Thus, if we multiply $a+b$
by $a-b$

$$\begin{array}{l} a^2+ab \\ \quad -ab-b^2 \end{array}$$

we obtain the product $a^2 \quad -b^2.$

EXAMPLES.

1. $(3a+2b)\,(3a-2b)=$
2. $(7ab+x)\,(7ab-x)=$
3. $(8a+7bc)\,(8a-7bc)=$
4. $(5a^2+6b^3)\,(5a^2-6b^3)=$
5. $(4a^2+3mx)\,(4a^2-3mx)=$
6. $(3a^2b+a^3)\,(3a^2b-a^3)=$
7. $(m+1)\,(m-1)=$
8. $(4+\frac{1}{3})\,(4-\frac{1}{3})=$

The pupil should be drilled upon examples like the preceding until he can produce the results mentally with as great facility as he could read them if exhibited upon paper, and without committing the common mistake of making the square of $a+b$ equal to a^2+b^2, or the square of $a-b$ equal to a^2-b^2.

The utility of these theorems will be the more apparent when they are applied to very complicated expressions. Frequent examples of their application will be seen hereafter.

CHAPTER V.

DIVISION.

70. *Division* is the converse of multiplication. In multiplication we determine the product arising from two given factors. In division we have the product and one of the factors given, and we are required to determine the other factor.

The *dividend* is the product of the divisor and quotient, the *divisor* is the given factor, and the *quotient* is the factor required to be found.

When the divisor and dividend are both monomials.

71. Since the product of the numbers denoted by a and b is denoted by ab, the quotient of ab divided by a is b; that is, $ab \div a = b$. Similarly, we have $abc \div a = bc$, $abc \div c = ab$, $abc \div ab = c$, etc. The division is more commonly denoted thus:

$$\frac{abc}{a}=bc, \quad \frac{abc}{b}=ac, \quad \frac{abc}{c}=ab,$$

$$\frac{abc}{bc}=a, \quad \frac{abc}{ac}=b, \quad \frac{abc}{ab}=c.$$

So, also, $12mn$ divided by $3m$ gives $4n$; for $3m$ multiplied by $4n$ makes $12mn$.

72. *Rule of Exponents in Division.*—Suppose we have a^5 to be divided by a^2. We must find a quantity which, multiplied by a^2, will produce a^5. We perceive that a^3 is such a quantity; for, according to *Art.* 58, in order to multiply a^3 by a^2, we add the exponents 2 and 3, making 5; that is, the exponent 3 of the quotient is found by subtracting 2, the exponent of the divisor, from 5, the exponent of the dividend.

Hence, in order to divide one power of any quantity by another power of the same quantity, *subtract the exponent of the divisor from the exponent of the dividend.*

73. *Proper Sign of the Quotient.*—The proper sign to be prefixed to a quotient may be deduced from the principles already established for multiplication. The product of the divisor and quotient must be equal to the dividend. Hence,

$$\text{because}\quad \left.\begin{aligned} +a\times(+b)&=+ab,\\ -a\times(+b)&=-ab,\\ +a\times(-b)&=-ab,\\ -a\times(-b)&=+ab,\end{aligned}\right\}\ \text{therefore}\ \left\{\begin{aligned} +ab\div(+b)&=+a.\\ -ab\div(+b)&=-a.\\ -ab\div(-b)&=+a.\\ +ab\div(-b)&=-a.\end{aligned}\right.$$

Hence, *if the dividend and divisor have like signs, the quotient will be positive; but if they have unlike signs, the quotient will be negative.*

74. Hence, for dividing one monomial by another, we have the following

RULE.

1. *Divide the coefficient of the dividend by the coefficient of the divisor, for a new coefficient.*

2. *To this result annex all the letters of the dividend, giving to each an exponent equal to the excess of its exponent in the dividend above that in the divisor.*

3. *If the dividend and divisor have like signs, prefix the plus sign to the quotient; but if they have unlike signs, prefix the minus sign.*

EXAMPLES.

1. Divide $20ax^3$ by $4x$. *Ans.* $5ax^2$.
2. Divide $25a^3xy^4$ by $-5ay^2$. *Ans.* $-5a^2xy^2$.
3. Divide $-72ab^5x^2$ by $12b^3x$. *Ans.* $-6ab^2x$.
4. Divide $-77a^3b^5c^6$ by $-11ab^3c^4$. *Ans.* $7a^2b^2c^2$.
5. Divide $48a^3b^3c^2d$ by $-12ab^2c$.
6. Divide $-150a^5b^8cd^3$ by $30a^3b^5d^2$.
7. Divide $-250a^7b^8x^3$ by $-5abx^3$.
8. Divide $272a^3b^4c^5x^6$ by $-17a^2b^3cx^4$.
9. Divide $-42a^6b^3c$ by $21ab^3c$.
10. Divide $-300a^3b^4x$ by $-50bx$.

75. *Value of the Symbol* a^0.—The rule given in *Art.* 72 conducts us in some cases to an expression of the form a^0. Let it be required to divide a^2 by a^2. According to the rule, the quotient will be a^{2-2}, or a^0. Now every number is contained in itself once; hence the value of the quotient must be unity; that is, $a^0=1$.

To demonstrate this principle generally, let a represent any quantity, and m the exponent of any power whatever. Then, by the rule of division,

$$a^m \div a^m = a^{m-m} = a^0.$$

But the quotient obtained by dividing any quantity by itself is unity; that is, $a^0=1$,

or *any quantity having a cipher for its exponent is equal to unity.*

76. *Signification of Negative Exponents.*—The rule given in *Art.* 72 conducts us in some cases to negative exponents. Thus, let it be required to divide a^3 by a^5. We are directed to subtract the exponent of the divisor from the exponent of the dividend. We thus obtain

$$a^{3-5}, \text{ or } a^{-2}.$$

But a^3 divided by a^5 may be written $\frac{a^3}{a^5}$; and, since the value of a fraction is not altered by dividing both numerator and denominator by the same quantity, this expression is equivalent to $\frac{1}{a^2}$.

Hence a^{-2} is equivalent to $\frac{1}{a^2}$.

So, also, if a^2 is to be divided by a^5, this may be written

$$\frac{a^2}{a^5}=\frac{1}{a^3}=a^{-3}.$$

In the same manner, we find

$$\frac{1}{a^m}=a^{-m};$$

that is, *any quantity having a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

77. Hence any factor may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator, by *changing the sign of its exponent.*

Thus, $\frac{a}{b}$ may be written ab^{-1},

$\frac{a^2}{b^2}$ " a^2b^{-2},

$\frac{ab^2}{c^2d^3}$ " $ab^2c^{-2}d^{-3}$;

that is, the denominator of a fraction may be entirely removed, and an *integral form* be given to any fractional expression.

This use of negative exponents must be understood simply as a convenient notation, and not as a method of actually destroying the denominator of a fraction.

78. *To divide a Polynomial by a Monomial.*—We have seen, *Art.* 60, that when a single term is multiplied into a polynomial, the former enters into *every term* of the latter.

Thus, $(a+b)m=am+bm$;

therefore $(am+bm)\div m=a+b$.

Hence, to divide a polynomial by a monomial, we have the following

RULE.

Divide each term of the dividend by the divisor, and connect the quotients by their proper signs.

EXAMPLES.

1. Divide $3x^3+6x^2+3ax-15x$ by $3x$. *Ans.* $x^2+2x+a-5$.
2. Divide $3abc+12abx-9a^2b$ by $3ab$. *Ans.* $c+4x-3a$.
3. Divide $40a^3b^3+60a^2b^2-17ab$ by $-ab$.
4. Divide $15a^2bc-10acx^2+5ac^2d^2$ by $-5a^2c$.
5. Divide $20x^5-35x^4-15x^3+75x^2$ by $-5x^2$.
6. Divide $6a^2x^4y^6-12a^3x^3y^6+15a^4x^5y^3$ by $3a^2x^2y^2$.
7. Divide $x^{n+1}-x^{n+2}+x^{n+3}-x^{n+4}$ by x^n.
8. Divide $12a^4y^6-16a^5y^5+20a^6y^4-28a^7y^3$ by $-4a^4y^3$.

79. *To divide one polynomial by another.*

Let it be required to divide

$$2ab+a^2+b^2 \text{ by } a+b.$$

The object of this operation is to find a third polynomial which, multiplied by the second, will reproduce the first.

It is evident that the dividend is composed of all the partial products arising from the multiplication of each term of the divisor by each term of the quotient, these products being added together and reduced. Hence, if we can discover a term of the dividend which is derived *without reduction* from the multiplication of a term of the divisor by a term of the quotient, then dividing this term by the corresponding term of the divisor, we shall be sure to obtain a term of the quotient.

But, from *Art.* 64, it appears that the term a^2, which contains the *highest exponent* of the letter a, is derived *without reduction* from the multiplication of the two terms of the divisor and quotient which are affected with the highest exponent of the same letter. Dividing the term a^2 by the term a of the divisor, we obtain a, which we are sure must be one term of the quotient sought. Multiplying each term of the divisor by a, and subtracting this product from the proposed dividend, the remainder may be regarded as the product of the divisor by the remaining terms of the quotient. We shall then obtain another term of the quotient by dividing that term of the remainder which is affected with the highest exponent of a by the term a of the divisor, and so on.

Thus we perceive that at each step we are obliged to search for that term of the dividend which is affected with the *highest exponent* of one of the letters, and divide it by that term of the divisor which is affected with the highest exponent of the same letter. We may avoid the necessity of *searching* for this term by arranging the terms of the divisor and dividend *in the order of the powers of one of the letters.*

The operation will then proceed as follows:

The arranged dividend is

$$\begin{array}{l|l} a^2+2ab+b^2 & a+b, \text{ the divisor.} \\ a^2+\ ab & \overline{a+b}, \text{ the quotient.} \\ \hline \quad\ ab+b^2, & \text{the first remainder.} \\ \quad\ ab+b^2 & \\ \hline \qquad\quad 0, & \text{remainder.} \end{array}$$

For convenience of multiplication, the divisor is written on the right of the dividend, and the quotient under the divisor.

80. Hence, to divide one polynomial by another, we have the following

RULE.

1. *Arrange both polynomials in the order of the powers of the same letter.*

2. *Divide the first term of the dividend by the first term of the divisor, for the first term of the quotient.*

3. *Multiply the whole divisor by this term, and subtract the product from the dividend.*

4. *Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient.*

5. *Multiply the whole divisor by this term, and subtract the product from the last remainder.*

6. *Continue the same operation until a remainder is found equal to zero, or one whose first term is not divisible by the first term of the divisor.*

When a remainder is found equal to zero, the division is said to be exact. When a remainder is found whose first term is not divisible by the first term of the divisor, the exact division is *impossible*. In such a case, the last remainder must be placed over the divisor in the form of a fraction, and annexed to the quotient.

EXAMPLES.

1. Divide $2a^2b+b^3+2ab^2+a^3$ by a^2+b^2+ab. *Ans.* $a+b$.

2. Divide $x^3-a^3+3a^2x-3ax^2$ by $x-a$. *Ans.* $x^2-2ax+a^2$.

3. Divide $a^6+x^6+2a^3x^3$ by a^2-ax+x^2.
Ans. $a^4+a^3x+ax^3+x^4$.

4. Divide $a^6-16a^3x^3+64x^6$ by $a^2-4ax+4x^2$.
5. Divide $a^4+6a^2x^2-4a^3x+x^4-4ax^3$ by $a^2-2ax+x^2$.
Ans. $a^2-2ax+x^2$.
6. Divide $32x^5+y^5$ by $2x+y$.
7. Divide $x^5+x^3y^2+xy^4-x^4y-x^2y^3-y^5$ by x^3-y^3.
8. Divide $x^4+x^3+5x-4x^2-3$ by x^2-2x-3.
9. Divide a^6-b^6 by $a^3+2a^2b+2ab^2+b^3$.
10. Divide x^6+1-2x^3 by x^2+1-2x.
11. Divide $x^4+y^4+x^2y^2$ by x^2+y^2+xy.
12. Divide $12x^4-192$ by $3x-6$.
Ans. $4x^3+8x^2+16x+32$.
13. Divide $6x^6-6y^6$ by $2x^2-2y^2$.
14. Divide $a^6+3a^2b^4-3a^4b^2-b^6$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^3+3a^2b+3ab^2+b^3$.
15. Divide $x^6-6x^4+9x^2-4$ by x^2-1.
16. Divide $a^4+a^3b-8a^2b^2+19ab^3-15b^4$ by $a^2+3ab-5b^2$.
17. Divide $x^3+y^3+3xy-1$ by $x+y-1$.
18. Divide $a^2b^2+2abc^2-a^2c^2-b^2c^2$ by $ab+ac-bc$.
19. Divide a^3-b^3 by $a-b$.
20. Divide a^4-b^4 by $a-b$.

81. Hitherto we have supposed the terms of the quotient to be obtained by dividing that term of the dividend which is affected with the *highest exponent* of a certain letter. But, from *Art.* 64, it appears that the term of the dividend affected with the *lowest* exponent of any letter is derived without reduction from the multiplication of a term of the divisor by a term of the quotient. Hence we may obtain a term of the quotient by dividing the term of the dividend affected with the *lowest exponent* of any letter by the term of the divisor containing the lowest exponent of the same letter; and we may even operate upon the highest and lowest exponents of a certain letter alternately in the same example.

82. *a^n-b^n is always divisible by $a-b$.* From the examples of *Art.* 80 we perceive that a^3-b^3 is divisible by $a-b$; and a^4-b^4 is divisible by $a-b$. We shall find the same to hold true, whatever may be the value of the exponents of the two letters; that is, *the difference of the same powers of any two quantities is always divisible by the difference of the quantities.*

Thus, let us divide a^5-b^5 by $a-b$:

$$\begin{array}{l|l} a^5-b^5 & a-b, \text{ divisor.} \\ \underline{a^5-a^4b} & a^4, \text{ partial quotient.} \\ \quad a^4b-b^5. & \end{array}$$

The first term of the quotient is a^4, and the first remainder is a^4b-b^5, which may be written

$$b(a^4-b^4).$$

Now if, after a division has been partially performed, the remainder is divisible by the divisor, it is obvious that the dividend is completely divisible by the divisor. But we have already found that a^4-b^4 is divisible by $a-b$; therefore a^5-b^5 is also divisible by $a-b$; and, in the same manner, it may be proved that a^6-b^6 is divisible by $a-b$, and so on.

83. To exhibit this reasoning in a more general form, let n represent any positive whole number whatever, and let us attempt to divide a^n-b^n by $a-b$. The operation will be as follows:

$$\begin{array}{l|l} a^n-b^n & a-b, \text{ divisor.} \\ \underline{a^n-ba^{n-1}} & a^{n-1}, \text{ quotient.} \end{array}$$

The first remainder is $ba^{n-1}-b^n$.

Dividing a^n by a, we have, by the rule of exponents, a^{n-1} for the quotient. Multiplying $a-b$ by this quantity, and subtracting the product from the dividend, we have for the first remainder $ba^{n-1}-b^n$, which may be written

$$b(a^{n-1}-b^{n-1}).$$

Now, if this remainder is divisible by $a-b$, it is obvious that the dividend is divisible by $a-b$; that is, *if the difference of the same powers of two quantities is divisible by the difference of the quantities, then will the difference of the powers of the next higher degree be divisible by that difference.*

Therefore, since a^4-b^4 is divisible by $a-b$, a^5-b^5 must be divisible by $a-b$; also a^6-b^6, and so on for any positive value of n.

The quotients obtained by dividing the difference of the same powers of two quantities by the difference of the quantities follow a simple law. Thus,

$$(a^2-b^2)\div(a-b)=a+b.$$
$$(a^3-b^3)\div(a-b)=a^2+ab+b^2.$$
$$(a^4-b^4)\div(a-b)=a^3+a^2b+ab^2+b^3.$$
$$(a^5-b^5)\div(a-b)=a^4+a^3b+a^2b^2+ab^3+b^4,$$
etc. etc. etc.

$$(a^n-b^n)\div(a-b)=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+a^2b^{n-3}+ab^{n-2}+b^{n-1}.$$

The exponents of a decrease by unity, while those of b increase by unity.

84. It may also be proved that *the difference of like even powers of any two quantities is always divisible by the sum of the quantities.*

Thus, $(a^2-b^2)\div(a+b)=a-b.$

$$(a^4-b^4)\div(a+b)=a^3-a^2b+ab^2-b^3.$$
$$(a^6-b^6)\div(a+b)=a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5,$$
etc. etc. etc.

Also, *the sum of like odd powers of any two quantities is always divisible by the sum of the quantities.*

Thus, $(a^3+b^3)\div(a+b)=a^2-ab+b^2.$

$$(a^5+b^5)\div(a+b)=a^4-a^3b+a^2b^2-ab^3+b^4.$$
$$(a^7+b^7)\div(a+b)=a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6,$$
etc. etc. etc.

The exponents of a and b follow the same law as in *Art.* 83, but the signs of the terms are alternately plus and minus.

85. *When exact division is impossible.*—One polynomial can not be divided by another polynomial containing *a letter which is not found in the dividend;* for it is impossible that one quantity multiplied by another which contains a certain letter should give a product *not containing* that letter.

A monomial is never divisible by a polynomial, because every polynomial multiplied by another quantity gives a product containing *at least two terms* not susceptible of reduction.

Yet a binomial may be divided by a polynomial containing *any number* of terms.

Thus, a^4-b^4 is divisible by $a^3+a^2b+ab^2+b^3$, and gives for a quotient $a-b$.

To resolve a Polynomial into Factors.

86. When a polynomial is capable of being resolved into factors, the factors can generally be discovered by inspection, or from the law of formation.

If all the terms of a polynomial have a common factor, that factor is a factor of the polynomial; and the other factor may be found by dividing the polynomial by the common factor.

EXAMPLES.

1. Resolve $3a^2b^2+3ab^3+3ab^2c$ into factors.
Ans. $3ab^2(a+b+c)$.
2. Resolve $5a^4b^2-10a^3b^3-5a^2b^4-5a^2b^2$ into factors.
Ans. $5a^2b^2(a^2-2ab-b^2-1)$.
3. Resolve $6a^2b^2c^3-12ab^2c^3mx-18ab^2c^3y$ into factors.
Ans. $6ab^2c^3(a-2mx-3y)$.
4. Resolve $7a^3b^2-7a^2b^3-7a^2b^2c$ into factors.
5. Resolve $8a^2bc+12ab^2c-16abc^2$ into factors.
6. Resolve $10ab^2cmx-5ab^2cy+5ab^2c$ into factors.

87. When two terms of a trinomial are *perfect squares*, and the third term is twice the product of their square roots, the trinomial will be the square of the sum or difference of these roots, *Arts.* 66 and 67, and may be resolved into factors accordingly.

EXAMPLES.

1. Resolve $a^2-2ab+b^2$ into factors. *Ans.* $(a-b)(a-b)$.
2. Resolve $a^2+4ab+4b^2$ into factors.
Ans. $(a+2b)(a+2b)$.
3. Resolve $a^2-6ab+9b^2$ into factors.

4. Resolve $9a^2-24ab+16b^2$ into factors.
5. Resolve $25a^4-60a^2b^3+36b^6$ into factors.
6. Resolve $4m^2n^2-4mn+1$ into factors.
7. Resolve $49a^4b^4-168a^3b^3+144a^2b^2$ into factors.
8. Resolve n^3+2n^2+n into three factors.
9. Resolve $16a^4b^2-24a^2bmx+9m^2x^2$ into factors.
10. Resolve $m^4n^2+2m^3n^3+m^2n^4$ into three factors.

88. If a binomial consists of two squares connected by the minus sign, it must be equal to the product of the sum and difference of the square roots of the two terms, *Art.* 69, and may be resolved into factors accordingly.

EXAMPLES.

1. Resolve $4a^2-9b^2$ into factors. *Ans.* $(2a+3b)(2a-3b)$.
2. Resolve $9a^2b^2-16a^2c^2$ into factors.
3. Resolve a^5x-9ax^3 into three factors.
4. Resolve a^4-b^4 into three factors.
5. Resolve a^6-b^6 into its factors.
6. Resolve a^8-b^8 into four factors.
7. Resolve $1-\frac{1}{25}$ into two factors.
8. Resolve $4-\frac{1}{49}$ into two factors.

89. If the two terms of a binomial are both powers of the same degree, it may generally be resolved into factors according to the principles of *Arts.* 82–84.

EXAMPLES.

1. Resolve a^3-b^3 into its factors. *Ans.* $(a^2+ab+b^2)(a-b)$.
2. Resolve a^3+b^3 into its factors.
3. Resolve a^6-b^6 into four factors.
4. Resolve a^3-8b^3 into its factors.
5. Resolve $8a^3-1$ into its factors.
6. Resolve $8a^3-8b^3$ into three factors.
7. Resolve $1+27b^3$ into its factors.
8. Resolve $8a^3+27b^3$ into its factors.
9. Resolve $a^{16}-b^{16}$ into five factors.

CHAPTER VI.

GREATEST COMMON DIVISOR.—LEAST COMMON MULTIPLE.

90. A *common divisor* of two quantities is a quantity which will divide them both without a remainder. Thus $2ab$ is a common divisor of $6a^2b^2x$ and $10a^3b^3y$.

91. A *prime factor* is one that can not be resolved into any other factors. It is, therefore, divisible only by itself and unity. Thus the quantity $2a^2-2ab$ is the product of the three prime factors 2, a, and $a-b$.

92. The *greatest common divisor* of two quantities is the greatest quantity which will divide each of them without a remainder. It is the continued product of all the prime factors which are common to both. The term *greatest* here refers to the *degree* of a quantity, or of its leading term, and not to its arithmetical value.

93. When both quantities can be resolved into prime factors by methods already explained, the greatest common divisor may be found by the following

RULE.

Resolve both quantities into their prime factors. The continued product of all those factors which are common to both, will be the greatest common divisor required.

EXAMPLES.

1. Find the greatest common divisor of $4a^2bx$ and $6ab^2x^3$.

Resolving into factors, we have

$$4a^2bx=2a\times2a\times b\times x.$$
$$6ab^2x^3=2a\times3b\times b\times x\times x\times x.$$

The common factors are $2a$, b, and x. Hence the greatest common divisor is $2abx$.

2. Find the greatest common divisor of $4am^2+4bm^2$ and $3an+3bn$.

Resolving into factors, we have

$$4am^2+4bm^2=2m\times 2m(a+b).$$
$$3an+3bn=3n(a+b).$$

Hence $a+b$ is the greatest common divisor.

3. Find the greatest common divisor of x^3-y^3 and x^2-y^2.

$$x^3-y^3=(x-y)(x^2+xy+y^2).$$
$$x^2-y^2=(x-y)(x+y).$$

Hence $x-y$ is the greatest common divisor.

4. Find the greatest common divisor of $35a^2bmx^2$ and $42am^2x^3$.

5. Find the greatest common divisor of $3a^2x-6abx+3b^2x$ and $4a^2y-4b^2y$.

6. Find the greatest common divisor of $9mx^2-6mx+m$ and $9nx^2-n$.

7. Find the greatest common divisor of $12a^2-36ab+27b^2$ and $8a^2-18b^2$.

94. When the given quantities can not be resolved into prime factors by inspection, the greatest common divisor may be found by applying the following principle:

The greatest common divisor of two quantities is the same with the greatest common divisor of the least quantity, and their remainder after division.

To prove this principle, let the greatest of the two quantities be represented by A, and the least by B. Divide A by B; let the entire part of the quotient be represented by Q, and the remainder by R. Then, since the dividend must be equal to the product of the divisor by the quotient, plus the remainder, we shall have $A=QB+R$.

Now every number which will divide B will divide QB; and every number which will divide R and QB will divide $R+QB$, or A. That is, every number which is a common divisor of B and R is a common divisor of A and B.

Again: every number which will divide A and B will divide A and QB; it will also divide $A-QB$, or R. That is, every number which is a common divisor of A and B is also a common divisor of B and R. Hence the greatest common divisor of A and B must be the same as the greatest common divisor of B and R.

95. *To find, then, the greatest common divisor* of two quantities, we divide the greater by the less; and the remainder, which is necessarily less than either of the given quantities, is, by the last article, divisible by the greatest common divisor. Dividing the preceding divisor by the last remainder, a still smaller remainder will be found, which is divisible by the greatest common divisor; and by continuing this process with each remainder and the preceding divisor, quantities smaller and smaller are found, which are all divisible by the greatest common divisor, until at length the greatest common divisor must be obtained. Hence we have the following

RULE.

Divide the greater quantity by the less, and the preceding divisor by the last remainder, till nothing remains; the last divisor will be the greatest common divisor.

When the remainders decrease to unity, the given quantities have *no* common divisor greater than unity, and are said to be *incommensurable*, or *prime* to each other.

EXAMPLES.

1. What is the greatest common divisor of 372 and 246?

```
               372 | 246
               246 | 1
          246 | 126, the first remainder.
          126 | 1
     126 | 120, the second remainder.
     120 | 1
120 |   6, the third remainder.
120 |  20
```

Here we have continued the operation of division until we obtain 0 for a remainder; the last divisor (6) is the greatest common divisor. Thus, 246 and 372, being each divided by 6, give the quotients 41 and 62, and these numbers are *prime* with respect to each other; that is, have no common divisor greater than unity.

2. What is the greatest common divisor of 336 and 720?
Ans. 48.

3. What is the greatest common divisor of 918 and 522?
Ans. 18.

96. *In applying this rule to polynomials* some modification may become necessary. It may happen that the first term of the arranged dividend is not divisible by the first term of the divisor. This may arise from the presence of a factor in the divisor which is not found in the dividend, and may therefore be suppressed. For, since the greatest common divisor of two quantities is only the product of their *common* factors, it can not be affected by a factor of the one quantity which *is not found* in the other.

We may therefore suppress in the first polynomial all the factors common to each of its terms. We do the same with the second polynomial; and if any factor suppressed is common to the two polynomials, we reserve it as one factor of the common divisor sought.

But if, after this reduction, the first term of the dividend, when arranged according to the powers of some letter, is not divisible by the first term of the arranged divisor, *we may multiply the dividend by any monomial factor which will render its first term divisible by the first term of the divisor.*

This multiplication will not affect the greatest common divisor, because we introduce into the dividend a factor which belongs *only to a part of the terms* of the divisor; for, by supposition, every factor common to all the terms has been suppressed.

97. The preceding principles are embodied in the following general

RULE.

1. *Arrange the two polynomials according to the powers of some letter; suppress all the monomial factors of each; and if any factor suppressed is common to the two polynomials, reserve it as one factor of the common divisor sought.*

2. *Multiply the first polynomial by such a monomial factor as will render its first term divisible by the first term of the second polynomial; then divide this result by the second polynomial, and continue the division till the first term of the remainder is of a lower degree than the first term of the divisor.*

3. *Take the second polynomial as a dividend, and the final remainder in the first operation as a divisor, and proceed as before, and so on till a remainder is found that will divide the preceding divisor. This remainder, multiplied by the common factors, if any, reserved at the beginning, will give the greatest common divisor.*

EXAMPLES.

1. Find the greatest common divisor of x^3+4x^2+5x+2 and x^2+5x+4.

$$\begin{array}{r|l} x^3+4x^2+5x+2 & x^2+5x+4 \\ \cline{2-2} x^3+5x^2+4x & x-1 \\ \hline -\ x^2+\ x+2 & \\ -\ x^2-5x-4 & \\ \hline 6x+6 & \end{array}$$

Suppressing the factor 6 in this remainder, we have $x+1$ for the next divisor.

$$\begin{array}{r|l} x^2+5x+4 & x+1 \\ \cline{2-2} x^2+\ x & x+4 \\ \hline 4x+4 & \\ 4x+4 & \end{array}$$

Here the division is exact; hence, by the rule, $x+1$ is the greatest common divisor sought.

2. Find the greatest common divisor of $6x^3-7ax^2-20a^2x$ and $6x^2+2ax-8a^2$.

Suppressing the factor 2 in the second polynomial, we proceed thus:

$$\begin{array}{l|l}
6x^3-7ax^2-20a^2x & 3x^2+ax-4a^2 \\ \cline{2-2}
6x^3+2ax^2-\ \ 8a^2x & 2x-3a \\ \cline{1-1}
\quad -9ax^2-12a^2x & \\
\quad -9ax^2-\ \ 3a^2x+12a^3 & \\ \cline{1-1}
\qquad\quad -\ \ 9a^2x-12a^3 &
\end{array}$$

Suppressing the factor $-3a^2$,

$$\begin{array}{l|l}
3x^2+\ \ ax-4a^2 & 3x+4a \\ \cline{2-2}
3x^2+4ax & x-a \\ \cline{1-1}
\quad -3ax-4a^2 & \\
\quad -3ax-4a^2 &
\end{array}$$

Hence $3x+4a$ is the greatest common divisor.

3. Find the greatest common divisor of $4a^3-2a^2-3a+1$ and $3a^2-2a-1$.

We first multiply the greater polynomial by 3, to render its first term divisible by the first term of the other polynomial.

$$\begin{array}{l|l}
12a^3-6a^2-9a+3 & 3a^2-2a-1 \\ \cline{2-2}
12a^3-8a^2-4a & 4a,\ +2 \\ \cline{1-1}
\qquad 2a^2-5a+3 & \\
 & \\
\qquad 6a^2-15a+\ \ 9 & \\
\qquad 6a^2-\ \ 4a-\ \ 2 & \\ \cline{1-1}
\qquad\quad -11a+11 &
\end{array}$$

Here we multiply the first remainder by 3, to render the first term divisible by the first term of the divisor. As the two partial quotients $4a$ and 2 have no connection, they are separated by a comma.

Rejecting the factor -11 from the second remainder, we proceed as follows:

$$\begin{array}{l|l}
3a^2-2a-1 & a-1 \\ \cline{2-2}
3a^2-3a & 3a+1 \\ \cline{1-1}
\qquad\ a-1 & \\
\qquad\ a-1 &
\end{array}$$

Hence $a-1$ is the greatest common divisor.

4. Find the greatest common divisor of $a^2-3ab+2b^2$ and $a^2-ab-2b^2$. *Ans.* $a-2b$.

5. Find the greatest common divisor of $a^3-a^2b+3ab^2-3b^3$ and $a^2-5ab+4b^2$. *Ans.* $a-b$.

6. Find the greatest common divisor of $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$. *Ans.* $3x-7$.

7. Find the greatest common divisor of

$$x^4-7x^3+8x^2+28x-48 \text{ and } x^3-8x^2+19x-14.$$

Ans. $x-2$.

98. *To find the greatest common divisor of three quantities.*—Find the greatest common divisor of the first and second, and then the greatest common divisor of this result and the third quantity. The last will be the greatest common divisor required.

EXAMPLES.

1. Find the greatest common divisor of $3a^2m^2$, $6b^2m^2$, and $12m^3x$. *Ans.* $3m^2$.

2. Find the greatest common divisor of $4x^3-21x^2+15x+20$, x^2-6x+8, and x^2-x-12. *Ans.* $x-4$.

3. Find the greatest common divisor of $6x^4+x^3-x$, $4x^3-6x^2-4x+3$, and $2x^3+x^2+x-1$. *Ans.* $2x-1$.

4. Find the greatest common divisor of $4x^4+9x^3+2x^2-2x-4$, $3x^3+5x^2-x+2$, and x^3+x^2-x+2. *Ans.* $x+2$.

LEAST COMMON MULTIPLE.

99. One quantity is a *multiple* of another when it can be divided by it without a remainder. Thus $5ab$ is a multiple of 5, also of a and of b. When one quantity is a multiple of another, the former must be equal to the product of the latter by some entire factor. Thus, if a is a multiple of b, then $a=mb$, where m is an entire number.

100. A *common multiple* of two or more quantities is one which can be divided by each separately without a remainder. Thus $20a^3b^2$ is a common multiple of $4ab$ and $5a^2b^2$.

101. The *least common multiple* of two or more quantities is the least quantity that can be divided by each without a remainder. Thus $12a^2$ is the least common multiple of $3a^2$ and $4a$.

102. It is obvious that the least common multiple of two or more quantities must contain *all the factors* of each of the quantities, and *no other factors*. Hence, when the given quantities can be resolved into prime factors, the least common multiple may be found by the following

RULE.

Resolve each of the quantities into its prime factors; take each factor the greatest number of times it enters any of the quantities; multiply together the factors thus obtained, and the product will be the least common multiple required.

EXAMPLES.

1. Find the least common multiple of $9x^2y$ and $12xy^2$.

Resolving into factors, we have

$$9x^2y=3\times 3xxy, \text{ and } 12xy^2=3\times 2\times 2xyy.$$

The factor 3 enters twice in the first quantity, also the factor 2 enters twice in the second; x twice in the first, and y twice in the second. Hence the least common multiple is

$$2\times 2\times 3\times 3xxyy, \text{ or } 36x^2y^2.$$

2. Find the least common multiple of $4a^2b^2$, $6a^2b$, and $10a^3x^2$.

We have
$$4a^2b^2=2\times 2aabb,$$
$$6a^2b=2\times 3aab,$$
$$10a^3x^2=2\times 5aaaxx.$$

Hence the least common multiple is

$$2\times 2\times 3\times 5aaabbxx, \text{ or } 60a^3b^2x^2.$$

3. Find the least common multiple of $a^2x-2abx+b^2x$ and a^2y-b^2y.

Here we have
$$a^2x-2abx+b^2x=(a-b)(a-b)x,$$
$$a^2y-b^2y=(a+b)(a-b)y.$$

Hence the least common multiple is

$$(a-b)(a-b)(a+b)xy, \text{ or } a^3xy-ab^2xy-a^2bxy+b^3xy.$$

4. Find the least common multiple of $5a^2b^2$, $10ab^3$, and $2abx$.
Ans. $10a^2b^3x$.

5. Find the least common multiple of $3ab^2$, $4ax^2$, $5b^2x$, and $6a^2x^2$.
Ans. $60a^2b^2x^2$.

6. Find the least common multiple of x^2-3x+2 and x^2-1.
Ans. $(x+1)(x-1)(x-2)$, or x^3-2x^2-x+2.

7. Find the least common multiple of a^3x+b^3x and $5a^2-5b^2$.
Ans. $5x(a+b)(a-b)(a^2-ab+b^2)$,
or $5a^4x-5a^3bx+5ab^3x-5b^4x$.

103. When the quantities can not be resolved into factors by any of the preceding methods, the least common multiple may be found by applying the following principles:

If two polynomials have no common divisor, their product must be their least common multiple; but if they have a common divisor, their product must contain the *second power* of this common divisor. Their least common multiple will therefore be obtained by dividing their product by their greatest common divisor. Hence, to find the least common multiple of two quantities, we have the following

RULE.

Divide the product of the two polynomials by their greatest common divisor; or divide one of the polynomials by the greatest common divisor, and multiply the other by the quotient.

EXAMPLES.

1. Find the least common multiple of $6x^2-x-1$ and $2x^2+3x-2$.

The greatest common divisor of the given quantities is $2x-1$. Hence the least common multiple is

$$\frac{(6x^2-x-1)(2x^2+3x-2)}{2x-1}, \text{ or } (2x^2+3x-2)(3x+1).$$

2. Find the least common multiple of x^3-1 and x^2+x-2.
Ans. $(x^3-1)(x+2)$.

3. Find the least common multiple of $x^3-9x^2+23x-15$ and x^2-8x+7. *Ans.* $(x^3-9x^2+23x-15)(x-7)$.

104. When there are *more than two polynomials*, find the least common multiple of any two of them; then find the least common multiple of this result, and a third polynomial; and so on to the last.

4. Find the least common multiple of a^2+2a-3, a^2-1, and $a-1$. *Ans.* $(a^2-1)(a+3)$.

5. Find the least common multiple of $4a^2+1$, $4a^2-1$, and $2a-1$. *Ans.* $16a^4-1$.

6. Find the least common multiple of a^3-a, a^3+1, and a^3-1. *Ans.* $a(a^6-1)$.

7. Find the least common multiple of $(x+2a)^3$, $(x-2a)^3$, and x^2-4a^2. *Ans.* $(x^2-4a^2)^3$.

CHAPTER VII.

FRACTIONS.

105. A *fraction* is a quotient expressed as described in *Art.* 71, by writing the divisor under the dividend with a line between them. Thus $\frac{a}{b}$ is a fraction, and is read a divided by b.

106. Every fraction is composed of two parts: the divisor, which is called the *denominator*, and the dividend, which is called the *numerator*.

107. An *entire quantity* is an algebraic expression which has no fractional part, as a^2-2ab.

An entire quantity may be regarded as a fraction whose denominator is unity. Thus, $a^2=\frac{a^2}{1}$.

108. A *mixed quantity* is an expression which has both entire and fractional parts. Thus $a^2+\frac{b}{c}$ is a mixed quantity.

109. *General Principles of Fractions.*—The following principles form the basis of most of the operations upon fractions:

1*st. In order to multiply a fraction by any number, we must multiply its numerator or divide its denominator by that number.*

Thus the value of the fraction $\frac{ab}{a}$ is b. If we multiply the numerator by a, we obtain $\frac{a^2b}{a}$, or ab; and if we divide the denominator of the same fraction by a, we obtain also ab; that is, the original value of the fraction, b, has been multiplied by a.

2*d. In order to divide a fraction by any number, we must divide its numerator or multiply its denominator by that number.*

Thus the value of the fraction $\frac{a^2b}{a}$ is ab. If we divide the numerator by a, we obtain $\frac{ab}{a}$, or b; and if we multiply the denominator of the same fraction by a, we obtain $\frac{a^2b}{a^2}$, or b; that is, the original value of the fraction ab has been divided by a.

3*d. The value of a fraction is not changed if we multiply or divide both numerator and denominator by the same number.*

Thus,
$$\frac{ab}{a}=\frac{abm}{am}=\frac{abmx}{amx}=b.$$

110. *The proper Sign of a Fraction.*—Each term in the numerator and denominator of a fraction has its own particular sign, and a sign is also written before the dividing line of a fraction. The relation of these signs to each other is determined by the principles already established for division. The sign prefixed to the numerator of a fraction affects merely the *dividend;* the sign prefixed to the denominator affects merely the *divisor;* but the sign prefixed to the dividing line of a fraction affects the quotient. The latter sign may be called the *apparent* sign of the fraction, while the *real* sign of the fraction is the sign of its numerical value when reduced.

The *real sign* of a fraction depends not merely upon its *apparent sign*, but also upon the signs of the numerator and denominator. From *Art.* 73, it follows that
$$\frac{ab}{a}=\frac{-ab}{-a}=+b,$$
and
$$\frac{-ab}{a}=\frac{ab}{-a}=-b.$$

Also, since a minus sign before the dividing line of a fraction shows that the quotient is to be subtracted, which is done by changing its sign, it follows that
$$-\frac{ab}{a}=-\frac{-ab}{-a}=-b,$$
and
$$-\frac{-ab}{a}=-\frac{ab}{-a}=+b.$$

Hence we see that of the three signs belonging to the numerator, denominator, and dividing line of a fraction, *any two may be changed from + to –, or from – to +, without affecting the real sign of the fraction.*

111. When the numerator or denominator of a fraction is a polynomial, it must be observed that by the sign of the numerator is to be understood the sign of the *entire numerator*, as distinguished from the sign of any one of its terms taken *singly*.

Thus, $-\frac{a+b+c}{x}$ is equivalent to $+\frac{-a-b-c}{x}$.

When *no* sign is prefixed either to the terms of a fraction or to its dividing line, plus is always to be understood.

Reduction of Fractions.

112. *To reduce a Fraction to its Lowest Terms.*—A fraction is in its lowest terms when the numerator and denominator contain no common factor; and since the value of a fraction is not changed if we divide both numerator and denominator by the same number (*Art.* 109), we have the following

RULE.

Divide both numerator and denominator by their greatest common divisor.

Or, *Cancel all those factors which are common to both numerator and denominator.*

EXAMPLES.

1. Reduce $\frac{a^2bc}{5a^2b^2}$ to its lowest terms.

We have $$\frac{a^2bc}{5a^2b^2}=\frac{c\times a^2b}{5b\times a^2b}.$$

Canceling the common factors a^2b, we have

$$\frac{a^2bc}{5a^2b^2}=\frac{c}{5b}.$$

2. Reduce $\frac{cx+x^2}{a^2c+a^2x}$ to its lowest terms.

We have $$\frac{cx+x^2}{a^2c+a^2x}=\frac{x(c+x)}{a^2(c+x)}=\frac{x}{a^2}.$$

3. Reduce $\frac{14a^2-7ab}{10ac-5bc}$ to its lowest terms. *Ans.* $\frac{7a}{5c}$.

4. Reduce $\frac{x^2-a^2}{x^4-a^4}$ to its lowest terms. *Ans.* $\frac{1}{x^2+a^2}$.

5. Reduce $\frac{2x^2-16x-6}{3x^2-24x-9}$ to its lowest terms. *Ans.* $\frac{2}{3}$.

6. Reduce $\frac{3x^2y+3xy^2}{3x^2+6xy+3y^2}$ to its lowest terms. *Ans.* $\frac{xy}{x+y}$.

7. Reduce $\frac{a^2-b^2}{a^2-2ab+b^2}$ to its lowest terms. $= \frac{a+b}{a-b}$.

8. Reduce $\frac{a^3-x^3}{a^2-2ax+x^2}$ to its lowest terms. $\frac{a^2+ax+x^2}{a-x}$.

9. Reduce $\frac{x^2-16}{x^2-x-20}$ to its lowest terms. $\frac{x-4}{x-5}$.

10. Reduce $\frac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$ to its lowest terms. $\frac{3x-1}{2x-1}$.

11. Reduce $\frac{6x^3-5x^2+4}{2x^3-x^2-x+2}$ to its lowest terms. $\frac{3x+2}{x+1}$.

12. Reduce $\frac{2x^3+9x^2+7x-3}{3x^3+5x^2-15x+4}$ to its lowest terms. $\frac{2x+3}{6x+8}$.

113. *To reduce a Fraction to an Entire or Mixed Quantity.*— When any term of the numerator is divisible by some term in the denominator, the division indicated by a fraction may be at least partially performed. Hence we have the following

RULE.

Divide the numerator by the denominator, continuing the operation as far as possible; then write the remainder, if any, over the denominator, and annex the fraction thus formed to the entire part.

EXAMPLES.

1. Reduce $\frac{ax-x^2}{x}$ to an entire quantity.

2. Reduce $\frac{ab-2a^2}{b}$ to a mixed quantity.

3. Reduce $\frac{a^2+x^2}{a-x}$ to a mixed quantity. *Ans.* $a+x+\frac{2x^2}{a-x}$.

4. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity.

5. Reduce $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

6. Reduce $\frac{8b^3-16b+7a^2b^2}{8b}$ to a mixed quantity.

7. Reduce $\frac{x^3-6x^2+11x-6}{x^2-3x+2}$ to an entire quantity.

8. Reduce $\frac{a^3+3a^2b+3ab^2+b^3}{a^2+2ab+b^2}$ to an entire quantity.

114. *To reduce a Mixed Quantity to the Form of a Fraction.*—This problem is the converse of the last, and we may proceed by the following

RULE.

Multiply the entire part by the denominator of the fraction; to the product add the numerator with its proper sign, and write the result over the denominator.

EXAMPLES.

1. Reduce $x+\frac{a^2-x^2}{x}$ to the form of a fraction. *Ans.* $\frac{a^2}{x}$.

2. Reduce $x+\frac{ax+x^2}{2a}$ to the form of a fraction.
Ans. $\frac{3ax+x^2}{2a}$.

3. Reduce $5+\frac{2x-7}{3x}$ to the form of a fraction.

4. Reduce $1+\frac{x-a-1}{a}$ to the form of a fraction.

5. Reduce $1+2x+\frac{x-3}{5x}$ to the form of a fraction.

6. Reduce $7+\frac{3b^2-8c^2}{a^2-b^2}$ to the form of a fraction.

7. Reduce $2a-7-\frac{4a^2-50}{2a+7}$ to the form of a fraction.

Ans. $\frac{1}{2a+7}$.

8. Reduce $(a-1)^2-\frac{(a-1)^2}{a}$ to the form of a fraction.

Ans. $\frac{(a-1)^3}{a}$.

115. *To reduce Fractions having Different Denominators to Equivalent Fractions having a Common Denominator:*

Suppose it is required to reduce the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$ to a common denominator. Since, by *Art.* 109, both terms of a fraction may be multiplied by the same quantity without changing its value, we may multiply both terms of each fraction by the product of the denominators of the other fractions, and we shall have

$$\frac{a}{b}=\frac{adn}{bdn},\ \frac{c}{d}=\frac{bcn}{bdn},\ \text{and}\ \frac{m}{n}=\frac{bdm}{bdn}.$$

The resulting fractions have the same value as the proposed fractions, and they have the common denominator bdn. Hence we have the following

RULE.

Multiply each numerator into all the denominators, except its own, for a new numerator, and all the denominators together for the common denominator.

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator. *Ans.* $\frac{ac}{bc}$, $\frac{ab+b^2}{bc}$.

2. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and $\frac{d}{4}$ to equivalent fractions having a common denominator.

3. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{4x}{5}$ to equivalent fractions having a common denominator.

4. Reduce $\frac{a}{2}$, $\frac{3x}{7}$, and $\frac{a+x}{a-x}$ to equivalent fractions having a common denominator.

5. Reduce $\frac{x}{3}$, $\frac{x+1}{5}$, and $\frac{1-x}{1+x}$ to equivalent fractions having a common denominator.

116. Fractions may always be reduced to a common denominator by the preceding rule; but if the denominators have any common factors, it will not be the *least* common denominator. The *least common denominator* of two or more fractions must be the least common multiple of their denominators.

Suppose it is required to reduce the fractions $\frac{2a}{3x^2}$ and $\frac{5b}{4x}$ to equivalent fractions having the least common denominator. The least common multiple of the denominators is $12x^2$. Multiply both terms of the first fraction by $\frac{12x^2}{3x^2}$, or 4, and both terms of the second fraction by $\frac{12x^2}{4x}$, or $3x$, and we shall have

$$\frac{8a}{12x^2} \text{ and } \frac{15bx}{12x^2},$$

which are equivalent to the given fractions, and have the least common denominator. Hence we deduce the following

RULE.

Find the least common multiple of all the denominators, and use this as the common denominator.

Divide this common denominator by each of the given denominators separately, and multiply each numerator by the corresponding quotient. The products will be the new numerators.

6. Reduce $\frac{2a}{3bc}$ and $\frac{bx}{6b^2c^2}$ to equivalent fractions having the least common denominator. *Ans.* $\frac{4ac}{6bc^2}$ and $\frac{x}{6bc^2}$.

7. Reduce $\frac{a+b}{a-b}$ and $\frac{c+d}{a^2-b^2}$ to equivalent fractions having the least common denominator. *Ans.* $\frac{(a+b)^2}{a^2-b^2}$ and $\frac{c+d}{a^2-b^2}$.

8. Reduce $\frac{a}{x^3}$, $\frac{m}{x^2}$, and $\frac{n}{x}$ to equivalent fractions having the least common denominator.

9. Reduce $\frac{9a}{8m}$, $\frac{7b}{36m}$, $\frac{11a}{28m}$, and $\frac{7(a+b)}{4m}$ to equivalent fractions having the least common denominator.

10. Reduce $\frac{2}{x}$, $\frac{3}{2x-1}$, and $\frac{2x-3}{4x^2-1}$ to equivalent fractions having the least common denominator.

Addition of Fractions.

117. The denominator of a fraction shows into how many parts a unit is to be divided, and the numerator shows how many of those parts are to be taken. Fractions can only be added when they are *like parts* of unity; that is, when they have a common denominator. In that case, the numerator of each fraction will indicate how many times the common fractional unit is repeated in that fraction, and the sum of the numerators will indicate how many times this result is repeated in the sum of the fractions. Hence we have the following

RULE.

Reduce the fractions to a common denominator; then add the numerators together, and write their sum over the common denominator.

If there are mixed quantities, we may add the entire and fractional parts separately.

EXAMPLES.

1. What is the sum of $\frac{x}{2}$ and $\frac{x}{3}$?

Reducing to a common denominator, the fractions become

$$\frac{3x}{6} \text{ and } \frac{2x}{6}.$$

Adding the numerators, we obtain $\frac{5x}{6}$.

It is plain that *three* sixths of x and *two* sixths of x make *five* sixths of x.

2. What is the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{m}{n}$?

Ans. $\frac{adn+bcn+bdm}{bdn}$.

3. What is the sum of $\frac{a}{a+b}$ and $\frac{b}{a-b}$?

4. What is the sum of $5x$, $\frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$?

5. What is the sum of $2a$, $3a+\frac{2x}{5}$, and $a+\frac{8x}{9}$?

Ans. $6a+\frac{58x}{45}$.

6. What is the sum of $a+x$, $\frac{a}{a-x}$, and $\frac{a-x}{a}$?

Ans. $a+x+2+\frac{x^2}{a^2-ax}$.

7. What is the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$? *Ans.* a.

8. What is the sum of $\frac{a}{2}$, $\frac{a-2m}{4}$, and $\frac{a+2m}{4}$?

9. What is the sum of $\frac{ma-b}{m+n}$ and $\frac{na+b}{m+n}$?

10. What is the sum of $\frac{x-n}{x+y+z}$, $\frac{y-z}{x+y+z}$, and $\frac{2z+n}{x+y+z}$?

11. What is the sum of $\frac{3y^2-2}{7y^2-5}$ and $\frac{7y^2+3}{4y^2-1}$?

12. What is the sum of $\frac{13a-29b}{5(a-b)}$, $-\frac{7b-21a}{5(a-b)}$, and $-\frac{9b-11a}{5(a-b)}$?

Ans. 9.

13. What is the sum of $\frac{1+x}{1-x}$, $\frac{1-x}{1+x}$, $-\frac{1-x+x^2}{1+x^2}$, $-\frac{1+x+x^2}{1-x^2}$, and -1?

Subtraction of Fractions.

118. Fractions can only be subtracted when they are *like parts* of unity; that is, when they have a common denominator. In that case, the difference of the numerators will indicate how many times the common fractional unit is repeated in the difference of the fractions. Hence we have the following

RULE.

Reduce the fractions to a common denominator; then subtract the numerator of the subtrahend from the numerator of the minuend, and write the result over the common denominator.

EXAMPLES.

1. From $\frac{2x}{3}$ subtract $\frac{3x}{5}$.

Reducing to a common denominator, the fractions become

$$\frac{10x}{15} \text{ and } \frac{9x}{15}.$$

Hence we have $$\frac{10x}{15}-\frac{9x}{15}=\frac{x}{15};$$

and it is plain that *ten* fifteenths of x, diminished by *nine* fifteenths of x, equals *one* fifteenth of x.

2. From $\frac{12x}{7}$ subtract $\frac{3x}{5}$.

3. From $\frac{9a-4x}{7}$ subtract $-\frac{5a-3x}{3}$.

It must be remembered that a minus sign before the dividing line of a fraction affects the *quotient* (*Art.* 111); and since a quantity is subtracted by changing its sign, the result of the subtraction in this case is

$$\frac{9a-4x}{7}+\frac{5a-3x}{3};$$

which fractions may be reduced to a common denominator, and the like terms united as in addition.

4. From $\frac{ax}{b-c}$ subtract $\frac{ax}{b+c}$. *Ans.* $\frac{2acx}{b^2-c^2}$.

5. From $2x+\frac{2+7x}{8}$ subtract $x-\frac{5x-6}{21}$. *Ans.* $\frac{355x-6}{168}$.

6. From $3x+\frac{x}{2b}$ subtract $x-\frac{x-a}{c}$.

7. From $\frac{a+b}{2}$ subtract $\frac{a-b}{2}$.

8. From $\frac{13a-5b}{4}$ subtract $\frac{7a-2b}{6}$. *Ans.* $\frac{25a-11b}{12}$.

9. From $\frac{6a-7b}{3a-2b}$ subtract $\frac{5a}{9b}$. *Ans.* $\frac{64ab-15a^2-63b^2}{27ab-18b^2}$.

10. From $\frac{(a+b)^2}{4ab}$ subtract unity. $\frac{(a-b)^2}{4ab}$.

11. From $\frac{2x}{11y}$ subtract $\frac{3x-8y}{7x-5y}$. $\frac{14x^2-43xy+88y^2}{11y(7x-5y)}$

12. From $\frac{a}{a-x}$ subtract $\frac{ax}{a^2-x^2}$.

Multiplication of Fractions.

119. Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

First let us multiply $\frac{a}{b}$ by c. According to the first principle of *Art.* 109, the product must be $\frac{ac}{b}$.

But the proposed multiplier was $\frac{c}{d}$; that is, we have used a multiplier d times too great. We must therefore divide the result by d; and, according to the second principle of *Art.* 109, we obtain

$$\frac{ac}{bd}; \text{ that is, } \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Hence we have the following

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

Entire and mixed quantities should first be reduced to fractional forms. Also, if there are any factors common to the numerator and denominator of the product, they should be canceled.

EXAMPLES.

1. Multiply $\frac{x}{6}$ by $\frac{2x}{9}$. *Ans.* $\frac{x^2}{27}$.

2. Multiply $\frac{x}{a}$ by $\frac{x+a}{a+b}$.

3. Multiply $b+\frac{bx}{a}$ by $\frac{a}{x}$.

4. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

5. Multiply $\frac{a^2+b^2}{a^2-b^2}$ by $\frac{a-b}{a+b}$. *Ans.* $\frac{a^2+b^2}{(a+b)^2}$.

6. Multiply together $\frac{x}{2}$, $\frac{4x}{5}$, and $\frac{10x}{21}$.

7. Multiply together $\frac{2x}{a}$, $\frac{3ab}{c}$, and $\frac{3ac}{2b}$.

8. Multiply together x, $\frac{x+1}{a}$, and $\frac{x-1}{a+b}$. *Ans.* $\frac{x^3-x}{a^2+ab}$.

9. Multiply $\frac{4ax}{(18m^2nx)(81m^2ny)}$ by $9m^2n$.

10. Multiply $\frac{mnx}{(a^2b^3c^5)(a^4b^3c^6)(a^7b^3c^4)}$ by $a^2b^2c^4$.

11. Multiply together $\frac{5a^3b^{12}}{7m^2n^4}$, $\frac{14a^9m}{25n^5b^{11}}$, $\frac{5n^{11}m^6}{6a^{15}b}$, and $\frac{6am}{b^3n}$.

Ans. $\frac{2m^6n}{a^2b^3}$.

12. Multiply together $\frac{13(a-b)}{7(m-n)}$, $\frac{5(x-y)}{39(a-b)}$, and $\frac{21(m-n)}{55(x-y)}$.

13. Multiply $\frac{3dn}{4ac}+\frac{3bm}{7}-\frac{5mn}{6bc}$ by $\frac{11abc}{13dmn}$.

14. Multiply together $\frac{3ax}{4by}$, $\frac{a^2-x^2}{c^2-x^2}$, $\frac{bc+bx}{a^2+ax}$, and $\frac{c-x}{a-x}$.

Ans. $\frac{3x}{4y}$.

15. Multiply together $\frac{1-x^2}{1+y}$, $\frac{1-y^2}{x+x^2}$, and $1+\frac{x}{1-x}$.

Ans. $\frac{1-y}{x}$.

16. Multiply $\frac{x(a-x)}{a^2+2ax+x^2}$ by $\frac{a(a+x)}{a^2-2ax+x^2}$.

17. Multiply $\frac{a^4-b^4}{a^2-2ab+b^2}$ by $\frac{a-b}{a^2+ab}$. *Ans.* $\frac{a^2+b^2}{a}$.

18. Multiply x^2-x+1 by $\frac{1}{x^2}+\frac{1}{x}+1$. *Ans.* $x^2+1+\frac{1}{x^2}$.

19. Multiply $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4b^2}{a^2-b^2}$ by $\frac{a+b}{2b}$. *Ans.* 2.

20. Multiply $\frac{x^2}{a^2}-\frac{x}{a}+1$ by $\frac{x^2}{a^2}+\frac{x}{a}+1$. *Ans.* $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$.

120. *Multiplication of Quantities affected with Negative Exponents.*—Suppose it is required to multiply $\frac{1}{a^3}$ by $\frac{1}{a^2}$.

According to the preceding article, the result must be $\frac{1}{a^5}$. But, according to *Art.* 76, $\frac{1}{a^3}$ may be written a^{-3}; $\frac{1}{a^2}$ may be written a^{-2}; and $\frac{1}{a^5}$ may be written a^{-5}.

Hence we see that $a^{-3}\times a^{-2}=a^{-5}$;
that is, the rule of *Art.* 58 is general, and *applies to negative as well as positive exponents.*

EXAMPLES.

1. Multiply $-x^{-2}$ by x^{-3}. *Ans.* $-x^{-5}$, or $-\frac{1}{x^5}$.
2. Multiply a^{-2} by $-a^3$.
3. Multiply a^{-3} by a^3.
4. Multiply a^{-m} by a^n.
5. Multiply a^{-m} by a^{-n}.
6. Multiply $(a-b)^5$ by $(a-b)^{-3}$.

Division of Fractions.

121. If the two fractions have the same denominator, then the quotient of the fractions will be the same as the quotient of their numerators. Thus it is plain that $\frac{3}{4}$ is contained in $\frac{9}{4}$ as often as 3 is contained in 9. If the two fractions have *not* the same denominator, we may perform the division after having first reduced them to a common denominator. Let it be required to divide $\frac{a}{b}$ by $\frac{c}{d}$.

Reducing to a common denominator, we have $\frac{ad}{bd}$ to be divided by $\frac{bc}{bd}$. It is now plain that the quotient must be represented by the division of ad by bc, which gives $\frac{ad}{bc}$;

a result which might have been obtained by inverting the terms of the divisor and multiplying by the resulting fraction; that is,

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

Hence we have the following

RULE.

Invert the terms of the divisor, and multiply the dividend by the resulting fraction.

Entire and mixed quantities should first be reduced to fractional forms.

EXAMPLES.

1. Divide $\frac{x}{3}$ by $\frac{2x}{9}$. *Ans.* $1\frac{1}{2}$.

2. Divide $\frac{2a}{b}$ by $\frac{4c}{d}$.

3. Divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$.

4. Divide $\frac{x+1}{6}$ by $\frac{2x}{3}$.

5. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

6. Divide $\frac{2ax+x^2}{c^3-x^3}$ by $\frac{x}{c-x}$. *Ans.* $\frac{2a+x}{c^2+cx+x^2}$.

7. Divide $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}-\frac{b}{a+b}$. *Ans.* Unity.

8. Divide $7a^2-3x+\frac{m}{n}$ by $b^2-\frac{a}{3}$.

Ans. $\frac{21a^2n-9nx+3m}{3b^2n-an}$.

9. Divide $15x^2-\frac{3ab}{5c}$ by $x-\frac{a-b}{c}$. *Ans.* $\frac{75cx^2-3ab}{5cx-5a+5b}$.

10. Divide $x^2+\frac{x^3}{a-b}$ by $\frac{ab}{a-b}-m$. *Ans.* $\frac{ax^2-bx^2+x^3}{ab-am+bm}$.

11. Divide $\frac{45(a-b)}{32(m+b)}$ by $\frac{27(a-b)}{128n(m+b)}$.

12. Divide $\frac{x^2}{y^3}+\frac{1}{x}$ by $\frac{x}{y^2}-\frac{1}{y}+\frac{1}{x}$. *Ans.* $\frac{x+y}{y}$.

13. Divide $\frac{x+2y}{x+y}+\frac{x}{y}$ by $\frac{x+2y}{y}-\frac{x}{x+y}$. *Ans.* Unity.

14. Divide $x^2+\frac{1}{x^2}+2$ by $x+\frac{1}{x}$. *Ans.* $\frac{x^2+1}{x}$.

15. Divide $a^2-b^2-c^2-2bc$ by $\frac{a+b+c}{a+b-c}$.

Ans. $a^2-b^2+c^2-2ac$.

122. *Division of Quantities affected with Negative Exponents.*— Suppose it is required to divide $\frac{1}{a^5}$ by $\frac{1}{a^3}$. According to the preceding article, we have

$$\frac{1}{a^5}\times\frac{a^3}{1}=\frac{a^3}{a^5}=\frac{1}{a^2}.$$

But, according to *Art.* 76, $\frac{1}{a^5}$ may be written a^{-5}; $\frac{1}{a^3}$ may be written a^{-3}; and $\frac{1}{a^2}$ may be written a^{-2}. Hence we see that

$$a^{-5}\div a^{-3}=a^{-2};$$

that is, the rule of *Art.* 72 is general, and *applies to negative as well as positive exponents.*

EXAMPLES.

1. Divide a^{-5} by $-a^{-2}$. *Ans.* $-a^{-3}$, or $-\frac{1}{a^3}$.
2. Divide $-a^2$ by a^{-1}.
3. Divide 1 by a^{-4}.
4. Divide $6a^n$ by $-2a^{-3}$.

5. Divide b^{m-n} by b^m.
6. Divide $12x^{-2}y^{-4}$ by $-4xy^2$.
7. Divide $(x-y)^{-4}$ by $(x-y)^{-6}$.

123. *The Reciprocal of a Fraction.*—According to the definition in *Art.* 34, the reciprocal of a quantity is the quotient arising from dividing a unit by that quantity. Hence the reciprocal of $\frac{a}{b}$ is

$$1 \div \frac{a}{b} = 1 \times \frac{b}{a} = \frac{b}{a};$$

that is, *the reciprocal of a fraction is the fraction inverted.*

Thus the reciprocal of $\frac{a}{b+x}$ is $\frac{b+x}{a}$; and the reciprocal of $\frac{1}{b+c}$ is $b+c$.

It is obvious that *to divide by any quantity is the same as to multiply by its reciprocal, and to multiply by any quantity is the same as to divide by its reciprocal.*

124. *How to simplify Fractional Expressions.*—The numerator or denominator of a fraction may be itself a fraction or a mixed quantity, as $\frac{2\frac{1}{2}}{\frac{3}{4}}$. In such cases we may regard the quantity above the line as a dividend, and the quantity below it as a divisor, and proceed according to *Art.* 121.

Thus, $$2\tfrac{1}{2} \div \tfrac{3}{4} = \tfrac{5}{2} \times \tfrac{4}{3} = \tfrac{10}{3} = 3\tfrac{1}{3}.$$

The most complex fractions may be simplified by the application of similar principles.

EXAMPLES.

1. Simplify the fraction $\dfrac{1+\frac{a}{b}}{1+\frac{b}{a}}$.

This expression is equivalent to $\frac{b+a}{b} \div \frac{a+b}{a}$,

or to $$\frac{b+a}{b} \times \frac{a}{a+b},$$ which is equal to $\frac{a}{b}$, *Ans.*

2. Simplify $\dfrac{1-\dfrac{a}{a-m}}{1-\dfrac{a}{a+m}}$. *Ans.* $-\dfrac{a+m}{a-m}$.

3. Find the value of the fraction $\dfrac{\frac{2}{3}\frac{4}{5}}{2\frac{2}{5}}$. *Ans.* $\frac{2}{7}$.

4. Find the value of the fraction $\dfrac{\dfrac{22abc}{39mnx}}{\dfrac{11ab}{3mx}}$.

5. Simplify $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}$. *Ans.* $\dfrac{ac-bd}{ac+bd}$.

6. Simplify $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}$. *Ans.* $\dfrac{a^2+x^2}{2ax}$.

7. Simplify $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}}\times\dfrac{m^2-n^2}{m^3+n^3}$. *Ans.* m.

8. Simplify $\dfrac{a}{b+\dfrac{c}{d+\dfrac{m}{n}}}$. *Ans.* $\dfrac{adn+am}{bdn+bm+cn}$.

CHAPTER VIII.

EQUATIONS OF THE FIRST DEGREE.

125. An *equation* is an expression of equality between two algebraic quantities. Thus $3x=2ab$ is an equation denoting that three times the quantity x is equal to twice the product of the quantities a and b.

126. The *first member* of the equation is the quantity on the left side of the sign of equality, and the *second member* is the quantity on the right of the sign of equality. Thus, in the preceding equation, $3x$ is the first member, and $2ab$ the second member.

127. The two members of an equation are not only equal numerically, but must have the *same essential sign.* If, in the preceding equation, x represents a negative quantity, then the first member is essentially negative, and the second member must also be negative; that is, either a or b must represent a negative quantity.

128. Equations are usually composed of certain quantities which are *known*, and others which are *unknown.* The known quantities are represented either by numbers, or by the first letters of the alphabet; the unknown quantities are usually represented by the last letters of the alphabet.

129. A *root* of an equation is the value of the unknown quantity in the equation; or it is any value which, being substituted for the unknown quantity, will satisfy the equation. For example, in the equation

$$3x-4=24-x,$$

suppose $x=7$. Substituting 7 for x, the first member becomes $3\times7-4$; that is, $21-4$, or 17; and the second member becomes $24-7$; that is, 17. Hence 7 is a root of the equation, because when substituted for x the two members are found to be equal.

130. A *numerical equation* is one in which all the known quantities are represented by figures; as, $x^3+4x^2=3x+12$.

131. A *literal equation* is one in which the known quantities are represented by letters, or by letters and numbers.

Thus $\left.\begin{array}{l} x^3+ax^2+bx=m, \\ \text{and} \quad x^4-3ax^3+5bx^2=16 \end{array}\right\}$ are literal equations.

132. The *degree of an equation* is denoted by the greatest number of unknown factors occurring in any term.

If the equation involves but one unknown quantity, its degree is denoted by the *exponent of the highest power* of this quantity in any term.

If the equation involves more than one unknown quantity, its degree is denoted by the *greatest sum* of the exponents of the unknown quantities in any term.

Thus $ax+b=cx+d$ is an equation of the *first degree*, and is sometimes called a *simple equation*.

$4x^2-2x=5-x^2$ and $7xy-4x+y=40$ are equations of the *second degree*, and are frequently called *quadratic equations*.

$x^3+ax^2=2b$ and $x^2+3xy^2+y=m$ are equations of the *third degree*, and are frequently called *cubic equations*.

So also we have equations of the fourth degree, sometimes called *bi-quadratic equations*; equations of the fifth degree, etc., up to the nth degree.

Thus $x^n+ax^{n-1}=b$ is an equation of the nth degree.

133. To *solve an equation* is to find the value of the unknown quantity, or to find a number which, being substituted for the unknown quantity in the equation, renders the first member identical with the second.

The difficulty of solving equations depends upon their degree, and the number of unknown quantities they contain.

134. *Axioms.*—The various operations which we perform upon equations, in order to deduce the value of the unknown quantities, are founded upon the following principles, which are regarded as self-evident.

1. If to two equal quantities the same quantity be *added*, the sums will be equal.
2. If from two equal quantities the same quantity be *subtracted*, the remainders will be equal.
3. If two equal quantities be *multiplied* by the same quantity, the products will be equal.
4. If two equal quantities be *divided* by the same quantity, the quotients will be equal.

135. *Transposition.*—Transposition is the process of changing a term from one member of an equation to the other without destroying the equality of the members.

Let it be required to solve the equation

$$x+a=b.$$

If from the two equal quantities $x+a$ and b we subtract the same quantity a, the remainders will be equal, according to the last article, and we shall have

$$x+a-a=b-a,$$

or

$$x=b-a.$$

Let it be required to solve the equation

$$x-a=b.$$

If to the two equal quantities $x-a$ and b the same quantity a be added, the sums will be equal, according to the last article, and we have $x-a+a=b+a$,

or

$$x=b+a.$$

136. Hence we perceive that *we may transpose any term of an equation from one member of the equation to the other, provided we change its sign.*

It is also evident that *we may change the sign of every term of an equation without destroying the equality;* for this is, in fact, the same thing as transposing *every term* in each member of the equation.

EXAMPLES.

In the following examples, transpose the unknown terms to the first member and the known terms to the second member.

1. $5x+12=3x+18$. *Ans.* $5x-3x=18-12$.
2. $4x-7=21-3x$. *Ans.* $4x+3x=21+7$.
3. $2x-15=-7x+30$. *Ans.* $2x+7x=30+15$.
4. $ax+bc=m-2x$. *Ans.* $ax+2x=m-bc$.
5. $4ax-b+2c=3x-2ab-3mx$.
Ans. $4ax-3x+3mx=b-2c-2ab$.
6. $4ab-ax-2c=bx-3m$. *Ans.* $ax+bx=4ab-2c+3m$.
7. $ab-cx-2mx=3ax-4b$.
Ans. $3ax+cx+2mx=ab+4b$.

137. *To clear an Equation of Fractions.*—Let the equation be $\frac{x}{a}=b$. If we multiply each of the equal quantities $\frac{x}{a}$ and b by the same quantity a, the products will be equal by *Art.* 134, and we shall have $x=ab$.

Suppose the equation is $\frac{x}{a}+\frac{x}{b}=m$.

If we multiply each of the members of the equation by a, we shall have

$$x+\frac{ax}{b}=am.$$

If we multiply each of the members of this equation by b, we shall have $bx+ax=abm$.

Hence, to clear an equation of fractions, we have the following

RULE.

Multiply each member of the equation by all the denominators.

EXAMPLES.

1. Clear the equation $\frac{x}{3}-\frac{x}{5}=\frac{3}{4}$ of fractions.
Ans. $20x-12x=45$.
2. Clear the equation $\frac{3x}{5}-\frac{2x}{3}=\frac{3}{7}$ of fractions.
Ans. $63x-70x=45$.

3. Clear the equation $\frac{2x}{7}-\frac{3x}{4}+\frac{x}{5}=6$ of fractions.

Ans. $40x-105x+28x=840$.

4. Clear the equation $\frac{x}{2}+\frac{x}{4}+\frac{x}{6}=10$ of fractions.

138. An equation may always be cleared of fractions by multiplying each member into all the denominators; but sometimes the same result may be attained by a less amount of multiplication. Thus, in the last example, the equation may be cleared of fractions by multiplying each term by 12 instead of $6\times4\times2$, and it is important to avoid all useless multiplication. In general, an equation may be cleared of fractions by multiplying each member by *the least common multiple of all the denominators.*

5. Clear the equation $\frac{2x}{5}+\frac{3x}{4}=\frac{7}{10}$ of fractions.

The least common multiple of all the denominators is 20. If we multiply each member of the equation by 20, we obtain

$$8x+15x=14.$$

The operation is effected by dividing the least common multiple by each of the denominators, and then multiplying the corresponding numerator, dropping the denominator.

6. Clear the equation $\frac{4x}{7}-\frac{3x}{14}=\frac{8}{21}$ of fractions.

7. Clear the equation $3x-\frac{x-4}{4}=\frac{1}{12}$ of fractions.

It should be remembered that when a fraction has the minus sign before it, this indicates that the fraction is to be *subtracted*, and the signs of the terms derived from its numerator must be changed, *Art.* 118.

Ans. $36x-3x+12=1$.

8. Clear the equation $\frac{a-x}{b}-\frac{3x-2b}{ab}=\frac{x+ab}{a^2}$.

Ans. $a^3-a^2x-3ax+2ab=bx+ab^2$.

139. *Solution of Equations.*—An equation of the first degree containing but one unknown quantity may be solved by transforming it in such a manner that the unknown quantity shall stand alone, constituting one member of an equation; the other member will then denote the value of the unknown quantity.

Let it be required to find the value of x in the equation

$$\frac{4x-2}{5}+\frac{5x}{8}=\frac{3x}{4}+5.$$

Clearing of fractions, we have

$$32x-16+25x=30x+200.$$

By transposition we obtain

$$32x+25x-30x=200+16.$$

Uniting similar terms, $27x=216$.

Dividing each member by 27, according to *Art.* 134, we have

$$x=8.$$

To verify this value of x, substitute it for x in the original equation, and we shall have

$$\frac{32-2}{5}+\frac{40}{8}=\frac{24}{4}+5,$$

or $$6+5=6+5;$$

that is, $$11=11,$$

an identical equation, which proves that we have found the correct value of x.

140. Hence we deduce the following

RULE.

1. *Clear the equation of fractions, and perform all the operations indicated.*

2. *Transpose all the terms containing the unknown quantity to one side, and all the remaining terms to the other side of the equation, and reduce each member to its most simple form.*

3. *Divide each member by the coefficient of the unknown quantity.*

There are various artifices which may sometimes be employed, by which the labor of solving an equation may be considerably abridged. These artifices can not always be reduced to general rules. If, however, any *reductions* can be made before clearing of fractions, it is generally best to make them; and if the equation contains several denominators, it is often best to multiply by the simpler denominators first, and then to effect any reductions which may be possible before getting rid of the remaining denominators. Sometimes considerable labor may be saved by simply *indicating* a multiplication during the first steps of the reduction, as we can thus more readily detect the presence of common factors (if there are any), which may be canceled. The discovery of these artifices will prove one of the most useful exercises to the pupil.

EXAMPLES.

1. Solve the equation

$$3x-\frac{8x+1}{7}=\frac{2x+9}{3}+4.$$

Clearing of fractions,

$$63x-24x-3=14x+63+84.$$

Transposing and reducing,

$$25x=150.$$

Dividing by 25, $x=6$.

To verify this result, put 6 in the place of x in the original equation.

Solve the following equations:

2. $3ax-4ab=2ax-6ac$. *Ans.* $x=4b-6c$.

3. $3x^2-10x=8x+x^2$. *Ans.* $x=9$.

4. $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}$. *Ans.* $x=\frac{d}{c}$.

5. $\frac{x-5}{4}+6x=\frac{284-x}{5}$. *Ans.* $x=9$.

6. $\frac{ab}{x}=bc+d+\frac{1}{x}$. *Ans.* $x=\frac{ab-1}{bc+d}$.

7. $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$. *Ans.* $x=7$.

8. $5ax-2b+4bx=2x+5c.$ *Ans.* $x=\frac{5c+2b}{5a+4b-2}.$

9. $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}.$ *Ans.* $x=5.$

10. $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}.$

11. $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}.$

12. $3x-a+cx=\frac{a+x}{3}-\frac{b-x}{a}.$ *Ans.* $x=\frac{4a^2-3b}{8a+3ac-3}.$

13. $\frac{3x}{a}-c+\frac{x}{b}=4x+\frac{2x}{d}.$ *Ans.* $x=\frac{abcd}{3bd+ad-4abd-2ab}.$

14. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2.$ *Ans.* $x=\frac{ac}{b}.$

15. $\frac{17-3x}{5}-\frac{4x+2}{3}=5-6x+\frac{7x+14}{3}.$

16. $x-\frac{3x-3}{5}-4=\frac{20x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}.$

17. $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}.$

18. $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}.$

19. $\frac{5}{6}ab+\frac{4}{5}ac-\frac{2}{3}cx=\frac{3}{4}ac+2ab-6cx.$ *Ans.* $x=\frac{70ab-3ac}{320c}.$

20. $\frac{1}{2}\left(x-\frac{a}{3}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=\frac{1}{3}\left(x-\frac{a}{4}\right).$ *Ans.* $x=\frac{8a}{25}.$

21. $\frac{7x+9}{4}-\left(x-\frac{2x-1}{9}\right)=7.$ *Ans.* $x=5.$

22. $\frac{x-1}{2}+\frac{x-2}{3}=\frac{x+3}{4}+\frac{x+4}{6}+1.$ *Ans.* $x=8\frac{3}{5}.$

23. $1-\frac{2}{3x}+4-\frac{5}{6x}=7-\frac{8}{9x}+10-\frac{11}{12x}.$ *Ans.* $x=\frac{11}{432}.$

24. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}.$ *Ans.* $x=11.$

25. $\frac{9x+4}{5x-48}+\frac{4x-19}{51}=\frac{5x+32}{17}-\frac{11x+13}{51}.$ *Ans.* $x=100.$

26. $\left(\frac{4}{x+2}+\frac{7}{x+3}\right)=\frac{37}{x^2+5x+6}.$ *Ans.* $x=1.$

27. $\left(\frac{1}{x-2}-\frac{1}{x-4}\right)=\left(\frac{1}{x-6}-\frac{1}{x-8}\right)$ *Ans.* $x=5.$

28. $\left(\frac{x}{a}+\frac{x}{b-a}\right)=\frac{a}{b+a}.$ *Ans.* $x=\frac{a^2(b-a)}{b(b+a)}.$

29. $\left(\left(x+\frac{5}{2}\right)\left(x-\frac{3}{2}\right)+\frac{3}{4}\right)=\left((x+5)(x-3)\right)$ *Ans.* $x=12.$

30. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=\frac{23}{x+1}+5.$ *Ans.* $x=3\frac{3}{8}.$

Solution of Problems.

141. A *problem* in Algebra is a question proposed requiring us to determine the value of one or more unknown quantities from given conditions.

142. The *solution* of a problem is the process of finding the value of the unknown quantity or quantities that will satisfy the given conditions.

143. The solution of a problem consists of two parts:

1st. The *statement*, which consists in expressing the conditions of the problem algebraically; that is, in translating the conditions of the problem from common into algebraic language, or *forming the equation.*

2d. The *solution of the equation.*

The second operation has already been explained, but the first is often more embarrassing to beginners than the second. Sometimes the conditions of a problem are expressed in a distinct and formal manner, and sometimes they are only *implied*, or are left to be inferred from other conditions. The former are called *explicit* conditions, and the latter *implicit* conditions.

144. It is impossible to give a general rule which will enable us to translate every problem into algebraic language, since the conditions of a problem may be varied indefinitely. The following directions may be found of some service:

Represent one of the unknown quantities by some letter or symbol, and then from the given conditions find an expression for each of the other unknown quantities, if any, involved in the problem.

Express in algebraic language the relations which subsist between the unknown quantities and the given quantities; or, by means of the algebraic signs, indicate the operations necessary to verify the value of the unknown quantity, if it was already known.

PROBLEMS.

Prob. 1. What number is that, to the double of which if 16 be added, the sum is equal to four times the required number?

Let x represent the number required.

The double of this will be $2x$.

This increased by 16 should equal $4x$.

Hence, by the conditions, $2x+16=4x$.

The problem is now translated into algebraic language, and it only remains to solve the equation in the usual way.

Transposing, we obtain

$$16=4x-2x=2x,$$

and $$8=x,$$

or $$x=8.$$

To verify this number, we have but to double 8, and add 16 to the result; the sum is 32, which is equal to four times 8, according to the conditions of the problem.

Prob. 2. What number is that, the double of which exceeds its half by 6?

Let x=the number required.

Then, by the conditions,

$$2x-\frac{x}{2}=6.$$

Clearing of fractions, $4x-x=12$,

or $$3x=12.$$

Hence $$x=4.$$

To verify this result, double 4, which makes 8, and diminish it by the half of 4, or 2; the result is 6, according to the conditions of the problem.

Prob. 3. The sum of two numbers is 8, and their difference 2. What are those numbers?

Let x=the least number.

Then $x+2$ will be the greater number.

The sum of these is $2x+2$, which is required to equal 8.

Hence we have $2x+2=8$.

By transposition, $2x=8-2=6$,

and $x=3$, the least number.

Also, $x+2=5$, the greater number.

Verification. $\left.\begin{array}{l}5+3=8\\5-3=2\end{array}\right\}$ according to the conditions.

The following is a generalization of the preceding Problem.

Prob. 4. The sum of two numbers is a, and their difference b. What are those numbers?

Let x represent the least number.

Then $x+b$ will represent the greater number.

The sum of these is $2x+b$, which is required to equal a.

Hence we have $2x+b=a$.

By transposition, $2x=a-b$,

or $x=\frac{a-b}{2}=\frac{a}{2}-\frac{b}{2}$, the less number.

Hence $x+b=\frac{a}{2}-\frac{b}{2}+b=\frac{a}{2}+\frac{b}{2}$, the greater number.

As these results are independent of any particular value attributed to the letters a and b, it follows that

Half the difference of two quantities, added to half their sum, is equal to the greater; and

Half the difference subtracted from half the sum is equal to the less.

The expressions $\frac{a}{2}+\frac{b}{2}$ and $\frac{a}{2}-\frac{b}{2}$ are called *formulas*, because they may be regarded as comprehending the solution of all questions of the *same kind;* that is, of all problems in which we have given the sum and difference of two quantities.

Thus, let $\left.\begin{array}{l} a=8 \\ b=2 \end{array}\right\}$ as in the preceding problem.

Then $\frac{a}{2}+\frac{b}{2}=\frac{8+2}{2}=5$, the greater number.

And $\frac{a}{2}-\frac{b}{2}=\frac{8-2}{2}=3$, the less number.

Given the sum of two numbers,		their difference		
	10;		6;	
	12;		2;	
	23;		11;	
	100;		50;	required the numbers.
	100;		1;	
	5;		$\frac{1}{2}$;	
	10;		$\frac{1}{2}$;	

Prob. 5. From two towns which are 54 miles distant, two travelers set out at the same time with an intention of meeting. One of them goes 4 miles and the other 5 miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then $4x$ will represent the number of miles one traveled, and $5x$ the number the other traveled; and since they meet, they must together have traveled the whole distance.

Consequently, $4x+5x=54$.

Hence $9x=54$,

or $x=6$.

Proof. In 6 hours, at 4 miles an hour, one would travel 24 miles; the other, at 5 miles an hour, would travel 30 miles. The sum of 24 and 30 is 54 miles, which is the whole distance.

This Problem may be generalized as follows:

Prob. 6. From two points which are a miles apart, two bodies move toward each other, the one at the rate of m miles per hour, the other at the rate of n miles per hour. In how many hours will they meet?

Let x represent the required number of hours.

Then mx will represent the number of miles one body moves, and nx the miles the other body moves, and we shall obviously have $mx+nx=a$.

Hence $$x=\frac{a}{m+n}.$$

This is a general formula, comprehending the solution of all problems of this kind. Thus,

$$\text{Let the distance}=\left\{\begin{array}{l}150;\\ \ 90;\\ 135;\\ 210;\end{array}\right. \text{one body moves}\left\{\begin{array}{l}\ 6;\\ \ 8;\\ 15;\\ 20;\end{array}\right. \text{the other}\left\{\begin{array}{ll}4 & \text{miles per hour.}\\ 7 & \text{“}\\ 12 & \text{“}\\ 15 & \text{“}\end{array}\right.$$

Required the time of meeting.

We see that an infinite number of problems may be proposed, all similar to Prob. 5; but they are all solved by the formula of Prob. 6. We also see what is necessary in order that the answers may be obtained in *whole numbers*. The given distance (a) must be exactly divisible by $m+n$.

Prob. 7. A gentleman, meeting three poor persons, divided 60 cents among them; to the second he gave twice, and to the third three times as much as to the first. What did he give to each?

Let x=the sum given to the first; then $2x$=the sum given to the second, and $3x$=the sum given to the third.

Then, by the conditions,

$$x+2x+3x=60.$$

That is, $$6x=60,$$

or $$x=10.$$

Therefore he gave 10, 20, and 30 cents to them respectively. The learner should verify this, and all the subsequent results.

The same problem generalized:

Prob. 8. Divide the number a into three such parts that the second may be m times, and the third n times as great as the first.

Ans. $\frac{a}{1+m+n}$; $\frac{ma}{1+m+n}$; $\frac{na}{1+m+n}$.

What is necessary in order that the preceding values may be expressed in whole numbers?

Prob. 9. A bookseller sold 10 books at a certain price, and afterward 15 more at the same rate. Now at the last sale he received 25 dollars more than at the first. What did he receive for each book? *Ans.* Five dollars.

The same Problem generalized:

Prob. 10. Find a number such that when multiplied successively by m and by n, the difference of the products shall be a.

$$Ans.\ \frac{a}{m-n}.$$

Prob. 11. A gentleman, dying, bequeathed 1000 dollars to three servants. A was to have twice as much as B, and B three times as much as C. What were their respective shares?

Ans. A received \$600, B \$300, and C \$100.

Prob. 12. Divide the number a into three such parts that the second may be m times as great as the first, and the third n times as great as the second.

$$Ans.\ \frac{a}{1+m+mn};\ \frac{ma}{1+m+mn};\ \frac{mna}{1+m+mn}.$$

Prob. 13. A hogshead which held 120 gallons was filled with a mixture of brandy, wine, and water. There were 10 gallons of wine more than there were of brandy, and as much water as both wine and brandy. What quantity was there of each?

Ans. Brandy 25 gallons, wine 35, and water 60 gallons.

Prob. 14. Divide the number a into three such parts that the second shall exceed the first by m, and the third shall be equal to the sum of the first and second.

$$Ans.\ \frac{a-2m}{4};\ \frac{a+2m}{4};\ \frac{a}{2}.$$

Prob. 15. A person employed four workmen, to the first of whom he gave 2 shillings more than to the second; to the second 3 shillings more than to the third; and to the third 4 shillings more than to the fourth. Their wages amount to 32 shillings. What did each receive?

Ans. They received 12, 10, 7, and 3 shillings respectively.

Prob. 16. Divide the number a into four such parts that the second shall exceed the first by m, the third shall exceed the second by n, and the fourth shall exceed the third by p.

Ans. The first, $\frac{a-3m-2n-p}{4}$; the second, $\frac{a+m-2n-p}{4}$;

the third, $\frac{a+m+2n-p}{4}$; the fourth, $\frac{a+m+2n+3p}{4}$.

Problems which involve several unknown quantities may often be solved by the use of a single unknown letter. Most of the preceding examples are of this kind. In general, when we have given the sum or difference of two quantities, both of them may be expressed by means of the same letter. For the difference of two quantities added to the less must be equal to the greater; and if one of two quantities be subtracted from their sum, the remainder will be equal to the other.

Prob. 17. At a certain election 36,000 votes were polled, and the candidate chosen wanted but 3000 of having twice as many votes as his opponent. How many voted for each?

Let x=the number of votes for the unsuccessful candidate; then $36{,}000-x$ = the number the successful one had, and $36{,}000-x+3000=2x$. *Ans.* 13,000 and 23,000.

Prob. 18. Divide the number a into two such parts that one part increased by b shall be equal to m times the other part.

Ans. $\frac{ma-b}{m+1}$; $\frac{a+b}{m+1}$.

Prob. 19. A train of cars, moving at the rate of 20 miles per hour, had been gone 3 hours, when a second train followed at the rate of 25 miles per hour. In what time will the second train overtake the first?

Let x=the number of hours the second train is in motion, and $x+3$=the time of the first train.

Then $25x$=the number of miles traveled by the second train, and $20(x+3)$=the miles traveled by the first train.

But at the time of meeting they must both have traveled the same distance.

Therefore $25x=20x+60$.

By transposition, $5x=60$,

and $x=12$.

Proof. In 12 hours, at 25 miles per hour, the second train goes 300 miles; and in 15 hours, at 20 miles per hour, the first train also goes 300 miles; that is, it is overtaken by the second train.

Prob. 20. Two bodies move in the same direction from two places at a distance of a miles apart; the one at the rate of n miles per hour, the other pursuing at the rate of m miles per hour. When will they meet?

Ans. In $\frac{a}{m-n}$ hours.

This Problem, it will be seen, is essentially the same as Prob. 10.

Prob. 21. Divide the number 197 into two such parts that four times the greater may exceed five times the less by 50.

Ans. 82 and 115.

Prob. 22. Divide the number a into two such parts that m times the greater may exceed n times the less by b.

Ans. $\frac{ma-b}{m+n}$; $\frac{na+b}{m+n}$.

When $n=1$, this Problem reduces to Problem 18.
When $b=0$, this Problem reduces to Problem 24.

Prob. 23. A prize of 2329 dollars was divided between two persons, A and B, whose shares were in the ratio of 5 to 12. What was the share of each?

Beginners almost invariably put x to represent one of the quantities sought in a problem; but a solution may often be very much simplified by pursuing a different method. Thus, in the preceding problem, we may put x to represent one fifth of A's share. Then $5x$ will be A's share, and $12x$ will be B's, and we shall have the equation

$$5x+12x=2329,$$

and hence $$x=137;$$

consequently their shares were 685 and 1644 dollars.

Prob. 24. Divide the number a into two such parts that the first part may be to the second as m to n.

Ans. $\frac{ma}{m+n}$; $\frac{na}{m+n}$.

Prob. 25. What number is that whose third part exceeds its fourth part by 16?

Let $12x=$the number.

Then $$4x-3x=16,$$

or $$x=16.$$
Therefore the number $=12\times16=192$.

Prob. 26. Find a number such that when it is divided successively by m and by n, the difference of the quotients shall be a.

Ans. $\frac{amn}{n-m}$.

Prob. 27. A gentleman has just 8 hours at his disposal; how far may he ride in a coach which travels 9 miles an hour, so as to return home in time, walking back at the rate of 3 miles an hour? *Ans.* 18 miles.

Prob. 28. A gentleman has just a hours at his disposal; how far may he ride in a coach which travels m miles an hour, so as to return home in time, walking back at the rate of n miles an hour?

Ans. $\frac{amn}{m+n}$ miles.

Prob. 29. A gentleman divides a dollar among 12 children, giving to some 9 cents each, and to the rest 7 cents. How many were there of each class?

Prob. 30. Divide the number a into two such parts that if the first is multiplied by m and the second by n, the sum of the products shall be b.

Ans. $\frac{b-na}{m-n}$; $\frac{ma-b}{m-n}$.

Prob. 31. If the sun moves every day 1 degree, and the moon 13, and the sun is now 60 degrees in advance of the moon, when will they be in conjunction for the first time, second time, and so on?

Prob. 32. If two bodies move in the same direction upon the circumference of a circle which measures a miles, the one at the rate of n miles per day, the other pursuing at the rate of m miles per day, when will they be together for the first time, second time, etc., supposing them to be b miles apart at starting?

Ans. In $\frac{b}{m-n}$, $\frac{a+b}{m-n}$, $\frac{2a+b}{m-n}$, etc., days.

It will be seen that this Problem includes Prob. 20.

Prob. 33. Divide the number 12 into two such parts that the difference of their squares may be 48.

Prob. 34. Divide the number a into two such parts that the difference of their squares may be b. *Ans.* $\frac{a^2-b}{2a}$; $\frac{a^2+b}{2a}$.

Prob. 35. The estate of a bankrupt, valued at 21,000 dollars, is to be divided among three creditors according to their respective claims. The debts due to A and B are as 2 to 3, while B's claims and C's are in the ratio of 4 to 5. What sum must each receive?

Prob. 36. Divide the number a into three parts, which shall be to each other as $m:n:p$.

Ans. $\frac{ma}{m+n+p}$; $\frac{na}{m+n+p}$; $\frac{pa}{m+n+p}$.

When $p=1$, Prob. 36 reduces to the same form as Prob. 8.

Prob. 37. A grocer has two kinds of tea, one worth 72 cents per pound, the other 40 cents. How many pounds of each must be taken to form a chest of 80 pounds, which shall be worth 60 cents?

Ans. 50 pounds at 72 cents, and 30 pounds at 40 cents.

Prob. 38. A grocer has two kinds of tea, one worth a cents per pound, the other b cents. How many pounds of each must be taken to form a mixture of n pounds, which shall be worth c cents?

Ans. $\frac{n(c-b)}{a-b}$ pounds at a cents,

and $\frac{n(a-c)}{a-b}$ pounds at b cents.

Prob. 39. A can perform a piece of work in 6 days; B can perform the same work in 8 days; and C can perform the same work in 24 days. In what time will they finish it if all work together?

Prob. 40. A can perform a piece of work in a days, B in b days, and C in c days. In what time will they perform it if all work together? *Ans.* $\frac{abc}{ab+ac+bc}$ days.

Prob. 41. There are three workmen, A, B, and C. A and B together can perform a piece of work in 27 days; A and C together in 36 days; and B and C together in 54 days. In what time could they finish it if all worked together?

A and B together can perform $\frac{1}{27}$ of the work in one day.
A and C " $\frac{1}{36}$ " "
B and C " $\frac{1}{54}$ " "

Therefore, adding these three results,

$$2A+2B+2C \text{ can perform } \tfrac{1}{27}+\tfrac{1}{36}+\tfrac{1}{54} \text{ in one day,}$$
$$=\tfrac{1}{12} \text{ in one day.}$$

Therefore, A, B, and C together can perform $\frac{1}{24}$ of the work in one day; that is, they can finish it in 24 days. If we put x to represent the time in which they would all finish it, then they would together perform $\frac{1}{x}$ part of the work in one day, and we should have

$$\tfrac{1}{27}+\tfrac{1}{36}+\tfrac{1}{54}=\tfrac{2}{x}.$$

Prob. 42. A and B can perform a piece of labor in a days; A and C together in b days; and B and C together in c days. In what time could they finish it if all work together?

Ans. $\dfrac{2abc}{ab+ac+bc}$ days.

This result, it will be seen, is of the same form as that of Problem 40.

Prob. 43. A broker has two kinds of change. It takes 20 pieces of the first to make a dollar, and 4 pieces of the second to make the same. Now a person wishes to have 8 pieces for a dollar. How many of each kind must the broker give him?

Prob. 44. A has two kinds of change; there must be a pieces of the first to make a dollar, and b pieces of the second to make the same. Now B wishes to have c pieces for a dollar. How many pieces of each kind must A give him?

Ans. $\dfrac{a(c-b)}{a-b}$ of the first kind; $\dfrac{b(a-c)}{a-b}$ of the second.

Prob. 45. Divide the number 45 into four such parts that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall all be equal.

In solving examples of this kind, several unknown quantities are usually introduced, but this practice is worse than superfluous. The four parts into which 45 is to be divided may be represented thus:

The first $=x-2$,
the second $=x+2$,
the third $=\frac{x}{2}$,
the fourth $=2x$;

for if the first expression be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the result in each case will be x. The sum of the four parts is $4\frac{1}{2}x$, which must equal 45.

Hence $x=10$.

Therefore the parts are 8, 12, 5, and 20.

Prob. 46. Divide the number a into four such parts that the first increased by m, the second diminished by m, the third multiplied by m, and the fourth divided by m, shall all be equal.

Ans. $\frac{ma}{(m+1)^2}-m;\ \frac{ma}{(m+1)^2}+m;\ \frac{a}{(m+1)^2};\ \frac{m^2a}{(m+1)^2}$.

Prob. 47. A merchant maintained himself for three years at an expense of $500 a year, and each year augmented that part of his stock which was not thus expended by one third thereof. At the end of the third year his original stock was doubled. What was that stock?

Prob. 48. A merchant supported himself for three years at an expense of a dollars per year, and each year augmented that part of his stock which was not thus expended by one third thereof. At the end of the third year his original stock was doubled. What was that stock?

Ans. $\frac{148a}{10}$.

Prob. 49. A father, aged 54 years, has a son aged 9 years. In how many years will the age of the father be four times that of the son?

Prob. 50. The age of a father is represented by a, the age of his son by b. In how many years will the age of the father be n times that of the son?

Ans. $\frac{a-nb}{n-1}$.

CHAPTER IX.

EQUATIONS OF THE FIRST DEGREE CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

145. If we have a single equation containing two unknown quantities, then for every value which we please to ascribe to one of the unknown quantities, we can determine the corresponding value of the other, and thus find as many *pairs of values* as we please which will satisfy the equation. Thus, let

$$2x+4y=16. \qquad (1.)$$

If $y=1$, we find $x=6$; if $y=2$, we find $x=4$, and so on; and each of these *pairs of values*, 1 and 6, 2 and 4, etc., substituted in equation (1), will satisfy it.

Suppose that we have another equation of the same kind, as, for example,

$$5x+3y=19. \qquad (2.)$$

We can also find as many pairs of values as we please which will satisfy this equation.

But suppose we are required to satisfy *both* equations with the *same set* of values for x and y; we shall find that there is only one value of x and one value of y. For, multiply equation (1) by 3, and equation (2) by 4, Axiom 3, and we have

$$6x+12y=48, \qquad (3.)$$

$$20x+12y=76. \qquad (4.)$$

Subtracting equation (3) from equation (4), Axiom 2, we have

$$14x=28; \qquad (5.)$$

whence

$$x=2. \qquad (6.)$$

Substituting this value of x in equation (1), we have

$$4+4y=16; \qquad (7.)$$

whence

$$y=3. \qquad (8.)$$

Thus we see that if *both* equations are to be satisfied, x *must* equal 2, and y *must* equal 3. Equations thus related are called simultaneous equations.

146. *Simultaneous equations* are those which must be satisfied by the same values of the unknown quantities.

When two or more simultaneous equations are given for solution, we must endeavor to deduce from them a *single equation* containing only one unknown quantity. We must therefore make one of the unknown quantities disappear, or, as it is termed, we must *eliminate* it.

147. *Elimination* is the operation of combining two or more equations in such a manner as to cause one of the unknown quantities contained in them to disappear.

There are three principal methods of elimination: 1st, by addition or subtraction; 2d, by substitution; 3d, by comparison.

148. *Elimination by Addition or Subtraction.*—Let it be proposed to solve the system of equations

$$5x+4y=35, \qquad (1.)$$
$$7x-3y=6. \qquad (2.)$$

Multiplying equation (1) by 3, and equation (2) by 4, we have

$$15x+12y=105, \qquad (3.)$$
$$28x-12y=24. \qquad (4.)$$

Adding (3) and (4), member to member (Axiom 1), we have

$$43x=129; \qquad (5.)$$

whence $$x=3. \qquad (6.)$$

We may now deduce the value of y by substituting the value of x in one of the original equations. Taking the first for example, we have $$15+4y=35;$$

whence $$4y=20,$$

and $$y=5.$$

149. In the same way, an unknown quantity may be eliminated from any two simultaneous equations. This method is expressed in the following

RULE.

Multiply or divide the equations, if necessary, in such a manner that one of the unknown quantities shall have the same coefficient in *both. Then subtract one equation from the other if the signs of these coefficients are alike, or add them together if the signs are unlike.*

In solving the preceding equations, we multiplied both members of each by the coefficient of the quantity to be eliminated in the other equation; but if the coefficients of the letter to be eliminated have any common factor, we may accomplish the same object by the use of smaller multipliers. In such cases, find the least common multiple of the coefficients of the letter to be eliminated, and divide this multiple by each coefficient; the quotients will be the *least multipliers* which we can employ.

150. *Elimination by Substitution.*—Take the same equations as before:

$$5x+4y=35, \qquad (1.)$$

$$7x-3y=6. \qquad (2.)$$

Finding from (1) the value of y in terms of x, we have

$$y=\frac{35-5x}{4}. \qquad (3.)$$

Substituting this value of y in (2), we have

$$7x-\frac{105-15x}{4}=6.$$

Clearing of fractions,

$$28x-105+15x=24;$$

whence

$$x=3.$$

Substituting this value of x in (3), we have

$$y=5.$$

The method thus exemplified is expressed in the following

RULE.

Find an expression for the value of one of the unknown quantities in one of the equations; then substitute this value for that quantity in the other equation.

151. *Elimination by Comparison.*—Take the same equations as before:

$$5x+4y=35, \qquad (1.)$$

$$7x-3y=6. \qquad (2.)$$

Derive from each equation an expression for y in terms of x, and we have

$$y=\frac{35-5x}{4}, \qquad (3.)$$

and

$$y=\frac{7x-6}{3}. \qquad (4.)$$

Placing these two values equal to each other, we have

$$\frac{7x-6}{3}=\frac{35-5x}{4}.$$

Clearing of fractions,

$$28x-24=105-15x;$$

whence

$$43x=129,$$

and

$$x=3.$$

Substituting this value of x in (3),

$$y=5.$$

The method thus exemplified is expressed in the following

RULE.

Find an expression for the value of the same unknown quantity in each of the equations, and form a new equation by placing these values equal to each other.

In the solution of simultaneous equations, either of the preceding methods can be used, as may be most convenient, and each method has its advantages in particular cases. Generally, however, the last two methods give rise to fractional expressions, which occasion inconvenience in practice, while the first method is not liable to this objection. When the coefficient of one of the unknown quantities in one of the equations is equal to unity, this inconvenience does not occur, and the method by substitution may be preferable; the first will, however, commonly be found most convenient.

EXAMPLES.

1. Given $\left\{\begin{array}{c} 11x+3y=100 \\ 4x-7y=4 \end{array}\right\}$ to find the values of x and y.

Ans. $x=8$; $y=4$.

2. Given $\left\{\begin{array}{l}\frac{x}{2}+\frac{y}{3}=7\\ \frac{x}{3}+\frac{y}{2}=8\end{array}\right\}$ to find x and y.

Ans. $x=6$; $y=12$.

3. Given $\left\{\begin{array}{l}\frac{x+2}{3}+8y=31\\ \frac{y+5}{4}+10x=192\end{array}\right\}$ to find x and y.

4. Given $\left\{\begin{array}{l}2y-\frac{x+3}{4}=7+\frac{3x-2y}{5}\\ 4x-\frac{8-y}{3}=24\frac{1}{2}-\frac{2x+1}{2}\end{array}\right\}$ to find x and y.

5. Given $\left\{\begin{array}{l}\frac{a}{x}+\frac{b}{y}=m\\ \frac{c}{x}+\frac{d}{y}=n\end{array}\right\}$ to find x and y.

Ans. $x=\frac{bc-ad}{nb-md}$; $y=\frac{bc-ad}{mc-na}$.

6. Given $\left\{\begin{array}{l}5x-7y=20\\ 9x-11y=44\end{array}\right\}$ to find x and y.

7. Given $\left\{\begin{array}{l}17x-13y=144\\ 23x+19y=890\end{array}\right\}$ to find x and y.

8. Given $\left\{\begin{array}{l}\frac{1}{x}=m-\frac{1}{y}\\ \frac{1}{y}=\frac{1}{x}-n\end{array}\right\}$ to find x and y.

Ans. $x=\frac{2}{m+n}$; $y=\frac{2}{m-n}$.

9. Given $\left\{\begin{array}{l}\frac{4x+81}{10y-17}=6\\ \frac{12x+97}{15y-17}=4\end{array}\right\}$ to find x and y.

Ans. $x=2\frac{1}{4}$; $y=3\frac{1}{5}$.

10. Given $\left\{\begin{array}{l}\frac{x+a}{n}+y-b=2a\\ x+a+\frac{y-b}{a}=1+na\end{array}\right\}$ to find x and y.

Ans. $x=na-a$; $y=a+b$.

11. Given $\left\{ \begin{array}{l} 1209\frac{1}{3}=60x+77y \\ 24x-35y=-152\frac{1}{3} \end{array} \right\}$ to find x and y.

Ans. $x=7\frac{3}{4}$; $y=9\frac{2}{3}$.

12. Given $\left\{ \begin{array}{l} \dfrac{13+x}{7}+\dfrac{3x-8y}{3}=x+y-5\frac{1}{3} \\ \dfrac{11-x}{2}+\dfrac{4x+8y-2}{9}=8-(y-x) \end{array} \right\}$ to find x and y.

Ans. $x=1$; $y=2$.

13. Given $\left\{ \begin{array}{l} \dfrac{5y}{6}-\dfrac{4y-19}{3}=\dfrac{x}{6}+\dfrac{20-2y}{3} \\ \dfrac{x+5y}{6}+5=\dfrac{2y+21}{3} \end{array} \right\}$ to find x and y.

Ans. $x=5$; $y=7$.

14. Given $\left\{ \begin{array}{l} \dfrac{13}{x+2y+3}=-\dfrac{3}{4x-5y+6} \\ \dfrac{3}{6x-5y+4}=\dfrac{19}{3x+2y+1} \end{array} \right\}$ to find x and y.

Ans. $x=7$; $y=8$.

15. Given $\left\{ \begin{array}{l} x^2-y^2=a \\ x-y=b \end{array} \right\}$ to find x and y.

$(x-y)(x+y)=a$
$b(x+y)=a$

Ans. $x=\dfrac{a+b^2}{2b}$; $y=\dfrac{a-b^2}{2b}$.

16. Given $\left\{ \begin{array}{l} \dfrac{2x}{3}-4+\dfrac{y}{2}+x=8-\dfrac{3y}{4}+\dfrac{1}{12} \\ \dfrac{y}{6}-\dfrac{x}{2}+2=\dfrac{1}{6}-2x+6 \end{array} \right\}$ to find x and y.

Ans. $x=2$; $y=7$.

17. Given $\left\{ \begin{array}{l} \dfrac{4x-3y-7}{5}=\dfrac{3x}{10}-\dfrac{2y}{15}-\dfrac{5}{6} \\ \dfrac{y-1}{3}+\dfrac{x}{2}-\dfrac{3y}{20}-1=\dfrac{y-x}{15}+\dfrac{x}{6}+\dfrac{1}{10} \end{array} \right\}$ to find x and y.

Ans. $x=3$; $y=2$.

Equations of the First Degree containing more than Two Unknown Quantities.

152. If we have *three* simultaneous equations containing three unknown quantities, we may, by the preceding methods, reduce two of the equations to one containing only two of the unknown quantities; then reduce the third equation and either of the former two to one containing the same two unknown quantities; and from the two equations thus obtained, the unknown quantities which they involve may be found. The third quantity may then be found by substituting these values in either of the proposed equations.

Take the system of equations

$$2x+3y+4z=16, \qquad (1.)$$
$$3x+2y-5z=8, \qquad (2.)$$
$$5x-6y+3z=6. \qquad (3.)$$

Multiplying (1) by 3, and (2) by 2, we have

$$6x+9y+12z=48, \qquad (4.)$$
$$6x+4y-10z=16. \qquad (5.)$$

Subtracting (5) from (4), $5y+22z=32.$ (6.)

Multiplying (1) by 5, and (3) by 2, we have

$$10x+15y+20z=80, \qquad (7.)$$
$$10x-12y+6z=12. \qquad (8.)$$

Subtracting (8) from (7), $27y+14z=68.$ (9.)

Multiplying (6) by 27, and (9) by 5, we have

$$135y+594z=864. \qquad (10.)$$
$$135y+70z=340. \qquad (11.)$$

Subtracting (11) from (10), $524z=524$;

whence $z=1.$

Substituting this value of z in (6),

$$5y+22=32;$$

whence $y=2.$

Substituting the values of y and z in (1),

$$2x+6+4=16;$$

whence $x=3.$

153. Hence, to solve three equations containing three unknown quantities, we have the following

RULE.

From the three equations deduce two containing only two unknown quantities; then from these two deduce one containing only one unknown quantity.

154. If we had *four* simultaneous equations containing four unknown quantities, we might, by the methods already explained, eliminate one of the unknown quantities. We should thus obtain three equations between three unknown quantities, which might be solved according to *Art.* 152. So, also, if we had *five* equations containing five unknown quantities, we might, by the same process, reduce them to four equations containing four unknown quantities, then to three, and so on. By following the same method, we might resolve a system of any number of equations of the first degree. Hence, if we have m equations containing m unknown quantities, we proceed by the following

RULE.

1*st. Combine successively any one of the equations with each of the others, so as to eliminate the same unknown quantity; there will result* $m-1$ *new equations, containing* $m-1$ *unknown quantities.*

2*d. Combine any one of these new equations with the others, so as to eliminate a second unknown quantity; there will result* $m-2$ *equations, containing* $m-2$ *unknown quantities.*

3*d. Continue this series of operations until there results a single equation containing but one unknown quantity, from which the value of this unknown quantity is easily deduced.*

4*th. Substitute this value for its equal in one of the equations containing two unknown quantities, and thus find the value of a second unknown quantity; substitute these values in an equation containing three unknown quantities, and find the value of a third; and so on, till the values of all are determined.*

Either of the unknown quantities may be selected as the one to be first eliminated. It is, however, generally best to begin with that which has the smallest coefficients; and if each of the unknown quantities is not contained in all the proposed equations, it is generally best to begin with that which is found in the least number of equations. Sometimes a solution may be very much abridged by the use of peculiar artifices, for which no general rules can be given.

EXAMPLES.

Solve the following groups of simultaneous equations:

1. $\begin{cases} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{cases}$ *Ans.* $\begin{cases} x=3. \\ y=7. \\ z=4. \end{cases}$

2. $\begin{cases} x+y=a \\ x+z=b \\ y+z=c \end{cases}$

Note. Take the sum of the three preceding equations.

3. $\begin{cases} x+y+z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{cases}$ *Ans.* $\begin{cases} x=8. \\ y=9. \\ z=12. \end{cases}$

4. $\begin{cases} x+\frac{1}{2}y+\frac{1}{3}z=32 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12 \end{cases}$

5. $\begin{cases} x+y-z=1320 \\ x-y+z=654 \\ -x+y+z=-12 \end{cases}$ *Ans.* $\begin{cases} x=987. \\ y=654. \\ z=321. \end{cases}$

6. $\begin{cases} x-y+z=6 \\ 3\frac{1}{2}x-4\frac{3}{4}y+5\frac{1}{2}z=32 \\ 10\frac{1}{2}x-9\frac{1}{2}y+11z=71 \end{cases}$ *Ans.* $\begin{cases} x=2. \\ y=4. \\ z=8. \end{cases}$

7. $\begin{cases} \dfrac{x}{5}+\dfrac{y}{7}+\dfrac{z}{9}=258 \\ \dfrac{x}{7}+\dfrac{y}{9}+\dfrac{z}{5}=304 \\ \dfrac{x}{9}+\dfrac{y}{5}+\dfrac{z}{7}=296 \end{cases}$ *Ans.* $\begin{cases} x=315. \\ y=630. \\ z=945. \end{cases}$

8. $\begin{cases} \frac{1}{x}+\frac{1}{y}=a \\ \frac{1}{x}+\frac{1}{z}=b \\ \frac{1}{y}+\frac{1}{z}=c \end{cases}$ *Ans.* $\begin{cases} x=\frac{2}{a+b-c}. \\ y=\frac{2}{a-b+c}. \\ z=\frac{2}{b+c-a}. \end{cases}$

9. $\begin{cases} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=a \\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}=b \\ -\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c \end{cases}$ *Ans.* $\begin{cases} x=\frac{2}{a+b}. \\ y=\frac{2}{a+c}. \\ z=\frac{2}{b+c}. \end{cases}$

10. $\begin{cases} \frac{12}{2x+3y}-\frac{7\frac{1}{2}}{3x+4z}=1 \\ \frac{30}{3x+4z}+\frac{37}{5y+9z}=3 \\ \frac{222}{5y+9z}-\frac{8}{2x+3y}=5 \end{cases}$ *Ans.* $\begin{cases} x=1. \\ y=2. \\ z=3. \end{cases}$

11. $\begin{cases} 2x+5y-7z=-288 \\ 5x-y+3z=227 \\ 7x+6y+z=297 \end{cases}$ *Ans.* $\begin{cases} x=13. \\ y=24. \\ z=62. \end{cases}$

12. $\begin{cases} \frac{x}{3}+\frac{y}{5}+\frac{2z}{7}=58 \\ \frac{5x}{4}+\frac{y}{6}+\frac{z}{3}=76 \\ \frac{x}{2}+\frac{3z}{8}+\frac{v}{5}=79 \\ y+z+v=248 \end{cases}$ *Ans.* $\begin{cases} x=12. \\ y=30. \\ z=168. \\ v=50. \end{cases}$

13. $\begin{cases} 7x-2z+3u=17 \\ 4y-2z+v=11 \\ 5y-3x-2u=8 \\ 4y-3u+2v=9 \\ 3z+8u=33 \end{cases}$ *Ans.* $\begin{cases} x=2. \\ y=4. \\ z=3. \\ u=3. \\ v=1. \end{cases}$

14. $\begin{cases} x+y+z+t+u=25 \\ x+y+z+u+v=26 \\ x+y+z+t+v=27 \\ x+y+t+u+v=28 \\ x+z+t+u+v=29 \\ y+z+t+u+v=30 \end{cases}$ *Ans.* $\begin{cases} x=3. \\ y=4. \\ z=5. \\ u=6. \\ t=7. \\ v=8. \end{cases}$

Note. Take the sum of these six equations.

Problems involving Equations of the First Degree with several Unknown Quantities.

Prob. 1. Find two numbers such that if the first be added to four times the second, the sum is 29; and if the second be added to six times the first, the sum is 36.

Prob. 2. If A's money were increased by 36 shillings, he would have three times as much as B; but if B's money were diminished by 5 shillings, he would have half as much as A. Find the sum possessed by each.

Prob. 3. A pound of tea and three pounds of sugar cost six shillings; but if sugar were to rise 50 per cent. and tea 10 per cent., they would cost seven shillings. Find the price of tea and sugar. *Ans.* Tea, 5*s.* per pound; Sugar, 4 pence.

Prob. 4. What fraction is that to the numerator of which if 4 be added the value is one half; but if 7 be added to the denominator, its value is one fifth? *Ans.* $\frac{5}{18}$.

Prob. 5. A certain sum of money, put out at simple interest, amounts in 8 months to $1488, and in 15 months it amounts to $1530. What is the sum and rate per cent.?

Prob. 6. A sum of money put out at simple interest amounts in m months to a dollars, and in n months to b dollars. Required the sum and rate per cent.

Ans. The sum is $\dfrac{na-mb}{n-m}$; the rate is $1200\times\dfrac{b-a}{na-mb}$.

Prob. 7. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4; but if the digits be inverted, and that number be divided by a number greater by two than the difference of the digits, the quotient is 14. Required the number.

Let x represent the left-hand digit, and y the right-hand digit.

Then, since x stands in the place of tens, the number will be represented by $10x+y$.

Hence, by the first condition,

$$\frac{10x+y}{x+y}=4;$$

by the second condition,

$$\frac{10y+x}{y-x+2}=14.$$

Whence $x=4$, $y=8$, and the required number is 48.

Prob. 8. A boy expends thirty pence in apples and pears, buying his apples at 4 and his pears at 5 for a penny, and afterward accommodates his friend with half his apples and one third of his pears for 13 pence. How many did he buy of each?

Prob. 9. A father leaves a sum of money to be divided among his children as follows: the first is to receive \$300 and the sixth part of the remainder; the second, \$600 and the sixth part of the remainder; and, generally, each succeeding one receives \$300 more than the one immediately preceding, together with the sixth part of what remains. At last it is found that all the children receive the same sum. What was the fortune left, and the number of children?

Ans. The fortune was \$7500, and the number of children 5.

Prob. 10. A sum of money is to be divided among several persons as follows: the first receives a dollars, together with the nth part of the remainder; the second, $2a$, together with the nth part of the remainder; and each succeeding one a dollars more than the preceding, together with the nth part of the remainder; and it is found at last that all have received the same sum. What was the amount divided, and the number of persons?

Ans. The amount was $a(n-1)^2$;
the number of persons $=n-1$.

Prob. 11. A wine-dealer has two kinds of wine. If he mixes

9 quarts of the poorer with 7 quarts of the better, he can sell the mixture at 55 cents per quart; but if he mixes 3 quarts of the poorer with 5 quarts of the better, he can sell the mixture at 58 cents per quart. What was the cost of a quart of each kind of wine?

Ans. 48 cents for the poorer, and 64 for the better.

Prob. 12. A person owes a certain sum to two creditors. At one time he pays them \$530, giving to one four elevenths of the sum which is due, and to the other \$30 more than one sixth of his debt to him. At a second time he pays them \$420, giving to the first three sevenths of what remains due to him, and to the other one third of what remains due to him. What were the debts?

Prob. 13. If A and B together can perform a piece of work in 12 days, A and C together in 15 days, and B and C in 20 days, how many days will it take each person to perform the same work alone?

This problem is readily solved by first finding in what time they could finish it if all worked together.

Prob. 14. If A and B together can perform a piece of work in a days, A and C together in b days, and B and C in c days, how many days will it take each person to perform the same work alone?

Ans. A in $\frac{2abc}{ac+bc-ab}$ days; B in $\frac{2abc}{ab+bc-ac}$ days;

C in $\frac{2abc}{ab+ac-bc}$ days.

Prob. 15. A merchant has two casks, each containing a certain quantity of wine. In order to have an equal quantity in each, he pours out of the first cask into the second as much as the second contained at first; then he pours from the second into the first as much as was left in the first; and then again from the first into the second as much as was left in the second, when there are found to be a gallons in each cask. How many gallons did each cask contain at first?

Ans. $\frac{11a}{8}$ and $\frac{5a}{8}$.

Prob. 16. A laborer is engaged for n days on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence. How many days did he work, and how many was he idle?

Ans. He worked $\frac{nq+a}{p+q}$ days, and was idle $\frac{np-a}{p+q}$ days.

Prob. 17. A certain number consisting of two digits contains the sum of its digits four times, and their product three times. What is the number?

Prob. 18. A father says to his two sons, of whom one was four years older than the other, In two years my age will be double the sum of your ages; but 6 years ago my age was 6 times the sum of your ages. How old was the father and each of the sons?

Ans. The father was 42, one son 11, and the other 7 years old.

Prob. 19. It is required to divide the number 96 into three parts such that if we divide the first by the second the quotient shall be 2, with 3 for a remainder; but if we divide the second by the third, the quotient shall be 4, with 5 for a remainder. What are the three parts? *Ans.* 61, 29, and 6.

Prob. 20. Each of seven baskets contains a certain number of apples. I transfer from the first basket to each of the other six as many apples as it previously contained; I next transfer from the second basket to each of the other six as many apples as it previously contained, and so on to the last basket, when it appeared that each basket contained the same number of apples, viz., 128. How many apples did each basket contain before the distribution?

Ans. The first 449, the second 225, the third 113, the fourth 57, the fifth 29, the sixth 15, and the seventh 8 apples.

155. When we have only one equation containing more than one unknown quantity, we can generally solve the equation in an infinite number of ways. For example, if a problem involving two unknown quantities (x and y) leads to the single equation

$$ax+by=c,$$

we may ascribe any value we please to x, and then determine the corresponding value of y. Such a problem is called indeterminate. *An indeterminate problem* is one which admits of an indefinite number of solutions.

156. If we had *two* equations containing *three* unknown quantities, we could, in the first place, eliminate one of the unknown quantities by means of the proposed equations, and thus obtain *one* equation containing *two* unknown quantities, which would be satisfied by an infinite number of systems of values. Therefore, in order that a problem may be *determinate*, its enunciation must contain as many different conditions as there are unknown quantities, and each of these conditions must be expressed by an *independent* equation.

157. Equations are said to be *independent* when they express conditions *essentially different*, and *dependent* when they express the *same* conditions under *different forms*.

Thus $\left\{ \begin{array}{c} x+y=7 \\ 2x+y=10 \end{array} \right\}$ are *independent* equations.

But $\left\{ \begin{array}{c} x+y=7 \\ 2x+2y=14 \end{array} \right\}$ are *not* independent, because the one may be deduced from the other.

158. If, on the contrary, the number of *independent* equations exceeds the number of unknown quantities, these equations will be *contradictory*.

For example, let it be required to find two numbers such that their sum shall be 8, their difference 2, and their product 20.

From these conditions we derive the following equations:

$$x+y=8,$$
$$x-y=2,$$
$$xy=20.$$

From the first two equations we find

$$x=5 \text{ and } y=3.$$

Hence the third condition, which requires that their product shall be equal to 20, *can not be fulfilled.*

CHAPTER X.

DISCUSSION OF PROBLEMS INVOLVING SIMPLE EQUATIONS.—INEQUALITIES.

159. To *discuss a problem* or an equation is to determine the values which the unknown quantities assume for particular hypotheses made upon the values of the given quantities, and to interpret the peculiar results obtained. We have seen that if the sum of two numbers is represented by a, and their difference by b, the greater number will be expressed by $\frac{a+b}{2}$, and the less by $\frac{a-b}{2}$. Here a and b may have any values whatever, and still these formulæ will always hold true. It frequently happens that, by attributing different values to the letters which represent known quantities, the values of the unknown quantities assume peculiar forms, which deserve consideration.

160. We may obtain *five species of values* for the unknown quantity in a problem of the first degree:

1st. Positive values.

2d. Negative values.

3d. Values of the form of zero, or $\frac{0}{A}$.

4th. Values of the form of $\frac{A}{0}$.

5th. Values of the form of $\frac{0}{0}$.

We will consider these five cases in succession.

161. 1st. *Positive values* are generally answers to problems in the sense in which they are proposed. Nevertheless, all positive values will not always satisfy the enunciation of a problem. For example, a problem may require an answer in *whole* *numbers*, in which case a fractional value of the unknown quantity is inadmissible. Thus, in Prob. 17, page 93, it is implied that the value of x must be a whole number, although this condition is not expressed in the equations. We might change the data of the problem, so as to obtain a fractional value of x, which would indicate an impossibility in the problem proposed. Problem 43, page 97, is of the same kind; also Prob. 7, page 109.

If the value obtained for the unknown quantity, even when positive, does not satisfy all the conditions of the problem, the problem is impossible in the form proposed.

162. 2d. *Negative values.*

Let it be proposed to find a number which, added to the number b, gives for a sum the number a. Let x denote the required number; then, by the conditions of the problem,

$$b+x=a;$$

whence $$x=a-b.$$

This formula will give the value of x corresponding to any assigned values of a and b.

For example, if $a=7$ and $b=4$,

then $$x=7-4=3,$$

a result which satisfies the conditions.

But suppose that $a=5$ and $b=8$,

then $$x=5-8=-3.$$

We thus obtain for x a *negative* value. How is it to be interpreted?

By referring to the problem, we see that it now reads thus: What number must be added to 8 in order that the sum may be 5? It is obvious that if the word *added* and the word *sum* are to retain their arithmetical meanings, the proposed problem is impossible. Nevertheless, if in the equation $8+x=5$ we substitute for $+x$ its value -3, it becomes

$$8-3=5,$$

an identical equation; that is, 8 diminished by 3 is equal to 5, or 5 may be regarded as the *algebraic sum* of 8 and -3.

The negative result, $x=-3$, indicates that the problem, in a strictly arithmetical sense, is impossible; but, taking this value of x with a contrary sign, we see that it satisfies the enunciation when modified as follows: What number must be *subtracted* from 8 in order that the *difference* may be 5? The second enunciation differs from the first only in this, that we put *subtract* for *add*, and *difference* for *sum*.

If we wish to solve this new equation directly, we shall have

$$8-x=5;$$

whence $$x=8-5, \text{ or } 3.$$

163. For another example, take Problem 50, page 98. The age of the father being represented by a, and that of the son by b, then $\frac{a-nb}{n-1}$ will represent the number of years before the age of the father will be n times that of the son.

Thus, suppose $a=54$, $b=9$, and $n=4$;

then $$x=\frac{54-36}{3}=\frac{18}{3}=6.$$

This value of x satisfies the conditions understood arithmetically; for if the father was 54 years old, and the son 9 years, then in 6 years more the age of the father will be 60 and the son 15; and we see that 60 is 4 times 15.

But suppose $a=45$, $b=15$, and $n=4$;

then $$x=\frac{45-60}{3}=\frac{-15}{3}=-5.$$

Here again we obtain a *negative* result. How are we to interpret it?

By referring to the problem, we see that the age of the son is already *more* than one fourth that of the father, so that the time required is already *past* by five years. The problem, if taken in a strictly arithmetical meaning, is impossible. But let us modify the enunciation as follows:

The age of the father is 45 years; the son's age is 15 years; how many years *since* the age of the father was four times that of his son?

The equation corresponding to this new enunciation is

$$15-x=\frac{45-x}{4};$$

whence $60-4x=45-x;$ and $x=5,$
a result which satisfies the modified problem taken in its arithmetical sense.

From this discussion we derive the following general principles:

1*st. A negative result found for the unknown quantity in a problem of the first degree indicates that the problem is impossible, if understood in its strict arithmetical sense.*

2*d. This negative value, taken with a contrary sign, may be regarded as the answer to a problem whose enunciation only differs from that of the proposed problem in this, that certain quantities which were* ADDED *should have been* SUBTRACTED, *and vice versa.*

164. In what case would the value of the unknown quantity in Prob. 20, page 94, be negative? *Ans.* When $n>m$.

Thus, let $m=20$, $n=25$, and $a=60$ miles;

then $$x=\frac{60}{20-25}=\frac{60}{-5}=-12.$$

To interpret this result, observe that it is impossible that the second train, which moves the slowest, should overtake the first. At the time of starting, the distance between them was 60 miles, and each subsequent hour the distance increases. If, however, we suppose the two trains to *have been* moving uniformly along an endless road, it is obvious that *at some former time* they must have been together.

This negative result indicates that the problem is impossible if understood in its strict arithmetical sense. But if the problem had been stated thus:

Two trains of cars, 60 miles apart, are moving in the same direction, the forward one 25 miles per hour, the other 20. *How long since they* were together?

The problem would have furnished the equation

$$25x=20x+60;$$

whence $$x=+12.$$

If we wish to include both of these cases in the same enunciation, the question should be, *Required the time of their being together*, leaving it uncertain whether the time was *past* or *future.*

EXAMPLES.

1. What number is that whose fourth part exceeds its third part by 16? *Ans.* -192.

How should the enunciation be modified in order that the result may be positive?

2. The sum of two numbers is 2, and their difference 8. What are those numbers? *Ans.* -3 and $+5$.

How should the enunciation be modified in order that both results may be positive?

3. What fraction is that from the numerator of which if 4 be subtracted the value is one half, but if 7 be subtracted from the denominator its value is one fifth? *Ans.* $\frac{-5}{-18}$.

How should the enunciation be modified in order that the problem may be possible in its arithmetical sense?

4. Find two numbers whose difference is 6, such that four times the less may exceed five times the greater by 12.

Ans. -42 and -36.

Change the enunciation of the problem so that these numbers, taken with the contrary sign, may be the answers to the modified problem.

165. 3d. We may obtain for the unknown quantity *values of the form of zero, or* $\frac{0}{A}$.

In what case would the value of the unknown quantity in Prob. 20, page 94, become zero, and what would this value signify?

Ans. This value becomes zero when $a=0$, which signifies that the two trains are together at the outset.

In what case would the value of the unknown quantity in Prob. 50, page 98, become zero, and what would this value signify?

Ans. When $a=nb$, which signifies that the age of the father is *now* n times that of the son.

In what case would the values of the unknown quantities in Prob. 38, page 96, become zero, and what would these values signify?

When a problem gives zero for the value of the unknown quantity, this value is sometimes applicable to the problem, and sometimes it indicates an impossibility in the proposed question.

166. 4th. We may obtain for the unknown quantity *values of the form of* $\frac{A}{0}$.

In what case does the value of the unknown quantity in Prob. 20, page 94, reduce to $\frac{A}{0}$, and how shall we interpret this result? *Ans.* When $m=n$.

On referring to the enunciation of the problem, we see that it is absolutely impossible to satisfy it; that is, there can be no point of meeting; for the two trains, being separated by the distance a, and moving equally fast, will always continue at the same distance from each other. The result $\frac{a}{0}$ may then be regarded as indicating an *impossibility.*

The symbol $\frac{a}{0}$ is sometimes employed to represent *infinity*, and for the following reason:

If the denominator of a fraction is made to *diminish*, while the numerator remains unchanged, the value of the fraction must *increase.*

For example, let $$m-n=0.01;$$

then $$x=\frac{a}{m-n}=\frac{a}{.01}=100a.$$

Let $$m-n=0.0001;$$

then $$x=\frac{a}{m-n}=\frac{a}{.0001}=10{,}000a.$$

Hence, if the difference in the rates of motion is not zero, the two trains must meet, and the time will become greater and greater as this difference is diminished. If, then, we suppose this difference to be *less than any assignable quantity*, the time represented by $\frac{a}{m-n}$ will be *greater than any assignable quantity.*

Hence we infer that every expression of the form $\frac{A}{0}$ found for the unknown quantity indicates the impossibility of satisfying the problem, at least in *finite* numbers.

In what case would the value of the unknown quantity in Prob. 10, page 92, reduce to the form $\frac{A}{0}$, and how shall we interpret this result?

167. The symbol 0, *called zero*, is sometimes used to denote the *absence* of value, and sometimes to denote a quantity *less than any assignable value.*

The symbol ∞, *called infinity*, is used to denote a quantity *greater than any assignable value.* A line produced beyond any assignable limit is said to be of infinite length; and time extended beyond any assignable limit is called infinite duration.

We have seen that when the denominator of the fraction $\frac{a}{m-n}$ becomes less than any assignable quantity, the value of the fraction becomes greater than any assignable quantity. Hence we conclude that $\frac{a}{0}=\infty$;

that is, *a finite quantity divided by zero is an expression for infinity.*

Also, if the denominator of a fraction be made to *increase* while the numerator remains unchanged, the value of the fraction must *diminish;* and when the denominator becomes *greater than any assignable quantity*, the value of the fraction must become *less than any assignable quantity.* Hence we conclude that

$$\frac{a}{\infty}=0;$$

that is, *a finite quantity divided by infinity is an expression for zero.*

168. 5th. We may obtain for the unknown quantity *values of the form of* $\frac{0}{0}$.

In what case does the value of the unknown quantity in Prob. 20, page 94, reduce to $\frac{0}{0}$, and how shall we interpret this result? *Ans.* When $a=0$, and $m=n$.

To interpret this result, let us recur to the enunciation, and observe that, since a is zero, both trains start from the same point; and since they both travel at the same rate, *they will always remain together;* and, therefore, the required point of meeting will be any where in the road traveled over. The problem, then, is entirely *indeterminate*, or admits of an infinite number of solutions; and the expression $\frac{0}{0}$ may represent any finite quantity.

We infer, therefore, that an expression of the form of $\frac{0}{0}$ found for the unknown quantity generally indicates that it may have any value whatever. In some cases, however, this value is subject to limitations.

In what case would the values of the unknown quantities in Prob. 44, page 97, reduce to $\frac{0}{0}$, and how would they satisfy the conditions of the problem? *Ans.* When $a=b=c$, which indicates that the coins are all of the same value. B might therefore be paid in either kind of coin; but there is a limitation, viz., that the value of the coins must be one dollar.

In what case do the values of the unknown quantities in Prob. 38, page 96, reduce to $\frac{0}{0}$, and how shall we interpret this result?

169. The expression $\frac{0}{0}$ may be conceived to result from a fraction whose numerator and denominator both diminish simultaneously, but in such a manner as to preserve the same *relative* value. If both numerator and denominator of a fraction are divided by the same quantity, its value remains unchanged. Hence, if $\frac{a}{b}$ represent any fraction, we may conceive both numerator and denominator to be divided by 10, 100, 1000, etc., until each becomes less than any assignable quantity, or 0. The fraction then reduces to the form of $\frac{0}{0}$, but the value of the fraction has throughout remained unchanged.

For example, we may suppose the numerator to represent the circumference of a circle, and the denominator to represent its diameter. The value of the fraction in this case is known to be 3.1416. If now we suppose the circle to diminish until it becomes a mere point, the circumference and diameter both become zero, but the value of the fraction has throughout remained the same. Hence, in this case, we have

$$\frac{0}{0}=3.1416.$$

Again, suppose the numerator to represent the area of a circle, and the denominator the area of the circumscribed square; then the value of the fraction becomes .7854. But this value remains unchanged, although the circle may be supposed to diminish until it becomes a mere point. Hence, in this case, we have

$$\frac{0}{0}=.7854.$$

Hence we conclude that the symbol $\frac{0}{0}$ *may represent any finite quantity.*

So, also, we may conceive both numerator and denominator of a fraction to be multiplied by 10, 100, 1000, etc., until each becomes greater than any assignable quantity; the fraction then reduces to the form of $\frac{\infty}{\infty}$. Hence we conclude that the symbol $\frac{\infty}{\infty}$ *may also represent any finite quantity.*

INEQUALITIES.

170. An *inequality* is an expression denoting that one quantity is greater or less than another. Thus $3x>2ab$ denotes that three times the quantity x is greater than twice the product of the quantities a and b.

171. In treating of inequalities, the terms greater and less must be understood in their algebraic sense; that is, a negative quantity standing alone is regarded as less than zero; and of two negative quantities, that which is *numerically the greatest* is considered as the *least;* for if from the same number we subtract successively numbers larger and larger, the remainders must continually *diminish.* Take any number, 5 for example, and from it subtract successively 1, 2, 3, 4, 5, 6, 7, 8, 9, etc., we obtain

$5-1,\ 5-2,\ 5-3,\ 5-4,\ 5-5,\ 5-6,\ 5-7,\ 5-8,\ 5-9$, etc.;

or, reducing, we have

$$4,\ 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4, \text{ etc.}$$

Hence we see that -1 should be regarded as less than zero; -2 less than -1; -3 less than -2, etc.

172. Two inequalities are said to subsist in *the same sense* when the greater quantity stands at the left in both, or at the right in both; and in *a contrary sense* when the greater quantity stands at the right in one and at the left in the other. Thus $9>7$ and $7>6$, or $5<8$ and $3<4$, are inequalities which subsist in the same sense; but the inequalities $10>6$ and $3<7$ subsist in a contrary sense.

173. *Properties of Inequalities.*—1st. *If the same quantity be added to or subtracted from each member of an inequality, the resulting inequality will always subsist in the same sense.*

Thus, $$8>3.$$

Adding 5 to each member, we have

$$8+5>3+5,$$

and subtracting 5 from each member, we have

$$8-5>3-5.$$

Again, take the inequality

$$-3<-2.$$

Adding 6 to each member, we have

$$-3+6<-2+6, \text{ or } 3<4;$$

and subtracting 6 from each member,

$$-3-6<-2-6, \text{ or } -9<-8.$$

174. Hence we conclude that we may *transpose* a term from one member of an inequality to the other, provided we change its sign.

Thus, suppose $a^2+b^2>3b^2-2a^2$.

Adding $2a^2$ to each member of the inequality, it becomes

$$a^2+b^2+2a^2>3b^2.$$

Subtracting b^2 from each member, we have

$$a^2+2a^2>3b^2-b^2,$$

or

$$3a^2>2b^2.$$

175. 2d. *If we add together the corresponding members of two or more inequalities which subsist in the same sense, the resulting inequality will always subsist in the same sense.*

Thus,

$$\begin{array}{r} 5>4 \\ 4>2 \\ 7>3 \\ \hline 16>9. \end{array}$$

Adding, we obtain $16>9$.

176. 3d. *If one inequality be subtracted from another which subsists in the same sense, the result will not always be an inequality subsisting in the same sense.*

Take the two inequalities $4<7$

and $2<3$

Subtracting, we have $4-2<7-3$, or $2<4$,

where the result is an inequality subsisting in the *same* sense.

But take $9<10$

and $6<\ 8$

Subtracting, we have $9-6>10-8$, or $3>2$,

where the result is an inequality subsisting in the *contrary* sense.

We should therefore avoid as much as possible the use of this transformation, or, when we employ it, determine in what sense the resulting inequality subsists.

177. 4th. *If we multiply or divide each member of an inequality by the same positive quantity, the resulting inequality will subsist in the same sense.*

Thus, if $a<b$,

then $ma<mb$,

and $\frac{a}{m}<\frac{b}{m}$.

Also, if $-a>-b$,

then $-ma>-mb$.

and $-\frac{a}{m}>-\frac{b}{m}$.

Hence an inequality may be cleared of fractions. Thus, suppose we have $\frac{a^2-b^2}{2d}>\frac{c^2-d^2}{3a}$.

Multiplying each member by $6ad$, it becomes

$$3a(a^2-b^2)>2d(c^2-d^2).$$

178. 5th. *If we multiply or divide each member of an inequality by the same negative number, the resulting inequality will subsist in the contrary sense.*

Take, for example, $8>7$.

Multiplying each member by -3, we have the opposite inequality $-24<-21$.

So, also, $15>12$.

Dividing each member by -3, we have

$$-5<-4.$$

Therefore, if we multiply or divide the two members of an inequality by an algebraic quantity, it is necessary to ascertain whether the multiplier or divisor is negative, for in this case the resulting inequality subsists in a contrary sense.

179. 6th. *If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

For to change all the signs is equivalent to multiplying each member of the inequality by -1.

180. *Reduction of Inequalities.*—The principles now established enable us to reduce an inequality so that the unknown quantity may stand alone as one member of the inequality. The other member will then denote one *limit* of the unknown quantity.

EXAMPLES.

1. Find a limit of x in the inequality

$$\frac{7}{6}-\frac{5x}{4}<\frac{95}{12}-2x.$$

Multiplying each member by 12, we have

$$14-15x<95-24x.$$

Transposing, $9x<81$.

Dividing, $x<9$.

2. $2x+\frac{x}{3}-8<6$. *Ans.* $x<6$.

3. $3x-2>\frac{5x}{2}-\frac{4}{5}$. *Ans.* $x>\frac{12}{5}$.

4. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{6}+\frac{x}{12}-7>9$.

5. $\frac{x}{6}-\frac{x-3}{15}<x-\frac{4x+1}{5}$. *Ans.* $x>4$.

6. Given $\left\{\begin{array}{l}\frac{3x}{2}+4x-8>3\\ 6x+\frac{5x-15}{3}<18\end{array}\right\}$ to find the limits of x.

Ans. $\left\{\begin{array}{l}x>2.\\ x<3.\end{array}\right.$

7. A man, being asked how many dollars he gave for his watch, replied, If you multiply the price by 4, and to the product add 60, the sum will exceed 256; but if you multiply the price by 3, and from the product subtract 40, the remainder will be less than 113. Required the price of the watch.

8. What number is that whose half and third part added together are less than 105; but its half diminished by its fifth part is greater than 33?

9. The double of a number diminished by 6 is greater than 22, and triple the number diminished by 6 is less than double the number increased by 10. Required the number.

CHAPTER XI.

INVOLUTION.

181. A *power* of a quantity is the product obtained by taking that quantity any number of times as a factor.

Thus the first power of 3 is 3;
the second power of 3 is 3×3, or 9;
the fourth power of 3 is $3 \times 3 \times 3 \times 3$, or 81, etc.

Involution is the process of raising a quantity to any power.

182. A power is indicated by means of an exponent. The *exponent* is a number or letter written a little above a quantity to the right, and shows how many times that quantity is taken as a factor.

Thus the first power of a is a^1, where the exponent is 1, which, however, is commonly omitted.

The second power of a is $a \times a$, or a^2, where the exponent 2 denotes that a is taken twice as a factor to produce the power aa.

The third power of a is $a \times a \times a$, or a^3, where the exponent 3 denotes that a is taken three times as a factor to produce the power aaa.

The fourth power of a is $a \times a \times a \times a$, or a^4.

Also the nth power of a is $a \times a \times a \times a$, etc., or a repeated as a factor n times, and is written a^n.

The second power is commonly called the *square*, and the third power the *cube*.

183. Exponents may be applied to polynomials as well as to monomials.

Thus $(a+b+c)^3$ is the same as

$$(a+b+c) \times (a+b+c) \times (a+b+c),$$

or the third power of the entire expression $a+b+c$.

Powers of Monomials.

184. Let it be required to find the third power or cube of $2a^3b^2$.

According to the rule for multiplication, we have

$(2a^3b^2)^3 = 2a^3b^2 \times 2a^3b^2 \times 2a^3b^2 = 2 \times 2 \times 2a^3a^3a^3b^2b^2b^2 = 8a^9b^6$.

In a similar manner any monomial may be raised to any power.

Hence, to raise a monomial to any power, we have the following

RULE.

Raise the numerical coefficient to the required power, and multiply the exponent of each of the letters by the exponent of the required power.

185. *Sign of the Power.*—With respect to the signs, it is obvious from the rules for multiplication that if the given monomial be positive, all of its powers are positive; but if the monomial be negative, its square is positive, its cube negative, its fourth power positive, and so on.

Thus
$$-a \times -a = +a^2,$$
$$-a \times -a \times -a = -a^3,$$
$$-a \times -a \times -a \times -a = +a^4,$$
$$-a \times -a \times -a \times -a \times -a = -a^5, \text{ etc.}$$

In general, *any even power of a negative quantity is positive, and every odd power negative; but all powers of a positive quantity are positive.*

EXAMPLES.

1. Find the square of $11a^3bcd^2$. *Ans.* $121a^6b^2c^2d^4$.
2. Find the square of $-18x^2yz^3$.
3. Find the cube of $7ab^2x^2$.
4. Find the cube of $-8xy^2z^3$.
5. Find the fourth power of $4ab^2c^3$.
6. Find the fourth power of $-5a^3b^2x$.
7. Find the fifth power of $2ab^3x^2$.
8. Find the fifth power of $-3ab^2x^4$.
9. Find the sixth power of $3ab^2x^3$.

10. Find the sixth power of $-2a^2b^3x^4$.
11. Find the seventh power of $2a^2x^3y$.
12. Find the mth power of ab^2x^3.

186. *Powers of Fractions.*—Let it be required to find the third power of $\frac{2ab^2}{3c}$.

From the rule for the multiplication of fractions, we have

$$\left(\frac{2ab^2}{3c}\right)^3 = \frac{2ab^2}{3c} \times \frac{2ab^2}{3c} \times \frac{2ab^2}{3c} = \frac{8a^3b^6}{27c^3}.$$

In a similar manner any fraction may be raised to any power. Hence, to raise a fraction to any power, we have the following

RULE.

Raise both numerator and denominator to the required power.

EXAMPLES.

1. Find the square of $\frac{3ab^2c^3}{7mn^2}$. *Ans.* $\frac{9a^2b^4c^6}{49m^2n^4}$.
2. Find the square of $-\frac{5a^2x^3}{4bc}$. *Ans.* $\frac{25a^4x^6}{16b^2c^2}$.
3. Find the cube of $\frac{-6ax^2y^3}{5mn}$.
4. Find the cube of $-\frac{ax^2}{9my^3}$.
5. Find the fourth power of $\frac{4ax^2y}{-3bm}$.
6. Find the fourth power of $-\frac{5ax^3z}{2bmn^2}$.
7. Find the fifth power of $-\frac{3ab^2x^3}{2my^2}$.
8. Find the fifth power of $\frac{4a^2x^2}{-3bm^3}$.
9. Find the sixth power of $-\frac{3a^2bxy^2}{2mn^2}$.

187. *Negative Exponents.*—The rule of *Art.* 184, for raising a monomial to any power, holds true *when the exponents of any of the letters are negative*, and also when *the exponent of the required power is negative.*

Let it be required to find the square of a^{-3}. This expression may be written $\frac{1}{a^3}$, which, raised to the second power, becomes $\frac{1}{a^6}$, or a^{-6}, the same result as would be obtained by multiplying the exponent -3 by 2.

Also, let it be required to find that power of $2am^2$ whose exponent is -3.

The expression $(2am^2)^{-3}$ may be written $\frac{1}{(2am^2)^3}$, which equals $\frac{1}{2^3a^3m^6}$. Transferring the factors to the numerator, we have
$$2^{-3}a^{-3}m^{-6}, \text{ or } \tfrac{1}{8}a^{-3}m^{-6}.$$

EXAMPLES.

Find the value of each of the following expressions.

1. $(3a^2b^{-4})^2$. *Ans.* $9a^4b^{-8}$.
2. $(7a^{-2}b^3c^{-4}x)^2$. *Ans.* $49a^{-4}b^6c^{-8}x^2$.
3. $(3ab^2x^{-1}y^{-2})^{-2}$. *Ans.* $\frac{1}{9}a^{-2}b^{-4}x^2y^4$.
4. $(-4a^2x^{-3}y^2)^{-2}$. *Ans.* $\frac{1}{16}a^{-4}x^6y^{-4}$.
5. $(-6ab^{-5}x^{-2})^3$.
6. $(-3a^2x^{-3}z^2)^{-3}$.
7. $(4a^{-3}bx^{-4})^{-3}$.
8. $(-3a^{-n}b^2x)^4$.
9. $(-4a^{-3}b^2x^{-4})^{-4}$.
10. $(-2ab^{-3}c^2x^{-4}y)^5$.

188. *Powers of Polynomials.*—A polynomial may be raised to any power by the process of continued multiplication. If the quantity be multiplied by itself, the product will be the second power; if the second power be multiplied by the original quantity, the product will be the third power, and so on. Hence we have the following

RULE.

Multiply the quantity by itself until it has been taken as a factor as many times as there are units in the exponent of the required power.

EXAMPLES.

1. Find the square of $2a+3b^2$. *Ans.* $4a^2+12ab^2+9b^4$.
2. Find the square of $a+m-n$.
3. Find the cube of $2a^2+3a-1$.
4. Find the cube of $a+\frac{1}{2}$.
5. Find the cube of $a+2b+3x$.
6. Find the fourth power of $a-b$.
7. Find the fourth power of $2a-3b$.
8. Find the fourth power of a^3+b^3.
9. Find the fifth power of $a-b$.
10. Find the square of $\frac{ax-by}{ay-bx}$. *Ans.* $\frac{a^2x^2-2abxy+b^2y^2}{a^2y^2-2abxy+b^2x^2}$.
11. Find the cube of $\frac{2a+3b}{m-n}$.
12. Find the cube of $\frac{a^2-b}{a-b^2}$.

189. *Square of a Polynomial.*—We have seen, *Art.* 66, that the square of a binomial may be formed without the labor of actual multiplication. The same principle may be extended to polynomials of any number of terms. By actual multiplication, we find the square of $a+b+c$ to be

$$a^2+b^2+c^2+2ab+2ac+2bc;$$

that is, *the square of a trinomial consists of the square of each term, together with twice the product of all the terms multiplied together two and two.*

In the same manner we find the square of $a+b+c+d$ to be

$$a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd;$$

that is, *the square of any polynomial consists of the square of each term, together with twice the sum of the products of all the terms multiplied together two and two.*

EXAMPLES.

1. Find the square of $a+b+c+d+x$.
2. Find the square of $a-b+c$.
3. Find the square of $1+2x+3x^2$.
4. Find the square of $1-x+x^2-x^3$.
5. Find the square of $a-2b+3ab-m$.
6. Find the square of $1-3x+3x^2-x^3$.
7. Find the square of $a-2b+3c-4d$.

In Chapter XVIII. will be given a method by which any power of a binomial may be obtained without the labor of multiplication.

CHAPTER XII.

EVOLUTION.

190. A *root* of a quantity is one of the equal factors which, multiplied together, will produce that quantity.

If a quantity be resolved into two equal factors, one of them is called the *square root.*

If a quantity be resolved into three equal factors, one of them is called the *cube root.*

If a quantity be resolved into four equal factors, one of them is called the *fourth root*, and so on.

191. *Evolution* is the process of extracting any root of a given quantity.

Evolution is indicated by the radical sign $\sqrt{}$.

Thus, $\sqrt{a}$ denotes the square root of a.

$\sqrt[3]{a}$ denotes the cube root of a.

$\sqrt[n]{a}$ denotes the nth root of a.

192. *Surds.*—When a root of an algebraic quantity which is required can not be exactly obtained, it is called an *irrational* or *surd* quantity.

Thus, $\sqrt[3]{a^2}$ is called a surd. $\sqrt{3}$ is also a surd, because the square root of 3 can not be expressed in numbers with perfect exactness.

A *rational* quantity is one which can be expressed in finite terms, and without any radical sign; as, a, $5a^2$, etc.

193. An *imaginary* root is one which can not be extracted on account of the *sign* of the given quantity. Thus the square root of -4 is impossible, because no quantity raised to an even power can produce a negative result.

A root which is not imaginary is said to be *real.*

Roots of Monomials.

194. According to *Art.* 184, in order to raise a monomial to any power, we raise the numerical coefficient to the required power, and multiply the exponent of each of the letters by the exponent of the power required. Hence, conversely, to extract any root of a monomial, we extract the root of the numerical coefficient, and divide the exponent of each letter by the index of the required root.

Thus the cube root of $64a^6b^3$ is $4a^2b$.

195. *Sign of the Root.*—We have seen, *Art.* 185, that all powers of a positive quantity are positive; but the even powers of a negative quantity are positive, while the odd powers are negative.

Thus $+a$, when raised to different powers in succession, will give $+a,\ +a^2,\ +a^3,\ +a^4,\ +a^5,\ +a^6,\ +a^7$, etc.

and $-a$, in like manner, will give

$$-a,\ +a^2,\ -a^3,\ +a^4,\ -a^5,\ +a^6,\ -a^7, \text{ etc.}$$

Hence it appears that if the root to be extracted be expressed by an *odd* number, the sign of the root will be the same as the sign of the proposed quantity. Thus, $\sqrt[3]{-a^3}=-a$; and $\sqrt[3]{+a^3}=+a$.

If the root to be extracted be expressed by an *even* number, and the quantity proposed be *positive*, the root may be either positive or negative. Thus, $\sqrt{a^2}=\pm a$.

If the root proposed to be extracted be expressed by an *even* number, and the sign of the proposed quantity be negative, the root can not be extracted, because no quantity raised to an even power can produce a negative result.

196. Hence, to extract any root of a monomial, we have the following

RULE.

1*st.* *Extract the required root of the numerical coefficient.*

2*d.* *Divide the exponent of each literal factor by the index of the required root.*

3*d*. *Every even root of a positive quantity must have the double sign* $\pm$, *and every odd root of any quantity must have the same sign as that quantity.*

From *Art.* 186, it is obvious that to extract any root of a fraction, we must divide the root of the numerator by the root of the denominator. Thus,

$$\sqrt[3]{\frac{a^3}{b^3}}=\frac{a}{b}; \text{ and } \sqrt[3]{-\frac{a^3}{b^3}}=-\frac{a}{b}.$$

EXAMPLES.

1. Find the square root of $64a^6b^4$. *Ans.* $\pm 8a^3b^2$.
2. Find the square root of $196a^2b^4c^6x^8$. *Ans.* $\pm 14ab^2c^3x^4$.
3. Find the square root of $225a^{2m}b^8x^6$.
4. Find the cube root of $64a^3b^6x^9$. *Ans.* $4ab^2x^3$.
5. Find the cube root of $-125a^3x^6y^9$. *Ans.* $-5ax^2y^3$.
6. Find the cube root of $-343a^6b^9x^{12}$.
7. Find the fourth root of $81a^4b^8$. *Ans.* $\pm 3ab^2$.
8. Find the fourth root of $256a^4b^{12}x^{16}$.
9. Find the fifth root of $-32a^5b^{10}x^{15}$. *Ans.* $-2ab^2x^3$.
10. Find the square root of $\frac{9a^2b^4}{16m^6x^{12}}$. *Ans.* $\pm\frac{3ab^2}{4m^3x^6}$.
11. Find the square root of $\frac{25a^4b^4x^6}{64a^2m^4y^4}$.
12. Find the cube root of $\frac{27a^3b^6}{8m^3x^9}$. *Ans.* $\frac{3ab^2}{2mx^3}$.
13. Find the cube root of $-\frac{125a^3b^9x^{12}}{216c^6z^9}$.
14. Find the fourth root of $\frac{256a^4x^8}{81b^8z^{16}}$.
15. Find the square root of $64a^{-2}b^{-4}x^4$. *Ans.* $\pm 8a^{-1}b^{-2}x^2$.
16. Find the cube root of $-512a^{-3}b^{-6}x^3$.
17. Find the fourth root of $256a^{-4}b^{-8}x^4$.
18. Find the fifth root of $-32a^{-10}b^{-15}x^5$. *Ans.* $-2a^{-2}b^{-3}x$.
19. Find the square root of $(a-b)^2x^6$. *Ans.* $\pm(a-b)x^3$.
20. Find the cube root of $(a+b)^3(x+y)^6$.

Square Root of Polynomials.

197. In order to discover a rule for extracting the square root of a polynomial, let us consider the square of $a+b$, which is $a^2+2ab+b^2$. If we arrange the terms of the square according to the dimensions of one letter, a, the first term will be the square of the first term of the root; and since, in the present case, the first term of the square is a^2, the first term of the root must be a.

Having found the first term of the root, we must consider the rest of the square, namely, $2ab+b^2$, to see how we can derive from it the second term of the root. Now this remainder may be put under the form $(2a+b)b$; whence it appears that we shall find the second term of the root if we divide the remainder by $2a+b$. The first part of this divisor, $2a$, is double of the first term already determined; the second part, b, is yet unknown, and it is necessary at present to leave its place empty. Nevertheless, we may commence the division, employing only the term $2a$; but as soon as the quotient is found, which in the present case is b, we must put it in the vacant place, and thus render the divisor complete.

The whole process, therefore, may be represented as follows:

$$\begin{array}{rl} & a^2+2ab+b^2\,(a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

If the square contained additional terms, we might continue the process in a similar manner. We may represent the first two terms of the root, $a+b$, by a single letter, m, and the remaining terms by c. The square of $m+c$ will be $m^2+2mc+c^2$. The square of the first two terms has already been subtracted from the given polynomial. If we divide the remainder by $2m$ as a partial divisor, we shall obtain c, which we place in the root, and also at the right of $2m$, to complete the divisor. We then multiply the complete divisor by c, and subtract the product from the dividend, and thus we continue until all the terms of the root have been obtained.

198. Hence we derive the following

RULE.

1*st. Arrange the terms according to the powers of some one letter; take the square root of the first term for the first term of the required root, and subtract its square from the given polynomial.*

2*d. Divide the first term of the remainder by twice the root already found, and annex the result both to the root and the divisor. Multiply the divisor thus completed by the last term of the root, and subtract the product from the last remainder.*

3*d. Double the entire root already found for a second divisor. Divide the first term of the last remainder by the first term of the second divisor for the third term of the root, and annex the result both to the root and to the second divisor, and proceed as before until all the terms of the root have been obtained.*

If the given polynomial be an exact square, we shall at last find a remainder equal to zero.

EXAMPLES.

1. Extract the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$\begin{array}{rl}
 & a^4-2a^3x+3a^2x^2-2ax^3+x^4\,(\,a^2-ax+x^2 \\
 & a^4 \\
2a^2-ax\,) & -2a^3x+3a^2x^2 \\
 & -2a^3x+\ \ a^2x^2 \\
2a^2-2ax+x^2\,) & 2a^2x^2-2ax^3+x^4 \\
 & 2a^2x^2-2ax^3+x^4
\end{array}$$

For verification, multiply the root a^2-ax+x^2 by itself, and we shall obtain the original polynomial.

2. Extract the square root of $a^2+2ab+2ac+b^2+2bc+c^2$.
3. Extract the square root of

$$10x^4-10x^3-12x^5+5x^2+9x^6-2x+1.$$

4. Extract the square root of

$$8ax^3+4a^2x^2+4x^4+16b^2x^2+16b^4+16ab^2x.$$

Ans. $2x^2+2ax+4b^2$.

5. Extract the square root of

$$15a^4b^2+a^6-6a^5b-20a^3b^3+b^6+15a^2b^4-6ab^5.$$

6. Extract the square root of $8ab^3+a^4-4a^3b+4b^4$.

7. Extract the square root of $4x^4+12x^3+5x^2-6x+1$.

Ans. $2x^2+3x-1$.

8. Extract the square root of

$$4x^4-12ax^3+25a^2x^2-24a^3x+16a^4.$$

Ans. $2x^2-3ax+4a^2$.

9. Extract the square root of

$$25x^4-30ax^3+49a^2x^2-24a^3x+16a^4.$$

Ans. $5x^2-3ax+4a^2$.

10. Extract the square root of $x^4-x^3+\frac{x^2}{4}+4x-2+\frac{4}{x^2}$.

Ans. $x^2-\frac{x}{2}+\frac{2}{x}$.

199. *When a Trinomial is a Perfect Square.*—The square of $a+b$ is $a^2+2ab+b^2$, and the square of $a-b$ is $a^2-2ab+b^2$. Hence the square root of $a^2\pm2ab+b^2$ is $a\pm b$; that is, a trinomial is a perfect square when two of its terms are squares, and the third is the double product of the roots of these squares.

Whenever, therefore, we meet with a quantity of this description, we may know that its square root is a binomial; and the root may be found by extracting the roots of the two terms which are complete squares, and connecting them by the sign of the other term.

EXAMPLES.

1. Find the square root of $4a^2+12ab+9b^2$. *Ans.* $2a+3b$.
2. Find the square root of $9a^2-24ab+16b^2$.
3. Find the square root of $9a^4-30a^3b+25a^2b^2$.
4. Find the square root of $4a^2+14ab+16b^2$, *if possible.*

No algebraic binomial can be a perfect square, for the square of a monomial is a monomial, and the square of a binomial necessarily consists of three distinct terms.

Square Root of Numbers.

200. The preceding rule is applicable to the extraction of the square root of numbers; for every number may be regarded as an algebraic polynomial, or as composed of a certain number of units, tens, hundreds, etc. Thus

529 is equivalent to $500+20+9$;

also, 841 " $800+40+1$.

If, then, 841 is the square of a number composed of tens and units, it must contain *the square of the tens, plus twice the product of the tens by the units, plus the square of the units.* But these three terms are blended together in 841, and hence arises the peculiar difficulty in determining its root. The following principles will, however, enable us to separate these terms, and thus detect the root.

201. 1*st. For every two figures of the square there will be one figure in the root, and also one for any odd figure.* Thus

the square of	1	is	1	the square of	1	is	1
"	9	"	81	"	10	"	1,00
"	99	"	98,01	"	100	"	1,00,00
"	999	"	99,80,01	"	1000	"	1,00,00,00

The smallest number consisting of two figures is 10, and its square is the smallest number of three figures. The smallest number of three figures is 100, and its square is the smallest number of five figures, and so on. Therefore the square root of every number composed of one or two figures will contain *one* figure; the square root of every number composed of three or four figures will contain *two* figures; of a number from five to six figures will contain *three* figures, and so on.

Hence, if we divide the number into periods of two figures, commencing at the units' place, the number of periods will indicate the number of figures in the square root.

202. 2*d. The first figure of the root will be the square root of the greatest square number contained in the first period on the left.*

For the square of tens can give no significant figure in the first right hand period, the square of hundreds can give no figure in the first two periods on the right, and the square of the highest figure in the root can give no figure except in the first period on the left.

Let it be required to extract the square root of 5329.

```
           53,29 ( 70+3, the root.
           49 00
140+3 )     4 29
            4 29
```

This number contains two periods, indicating that there will be two places in the root. Let $a+b$ denote the root, where a is the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of 10, which has its square less than 5300; this is found to be 70. Subtract a^2, that is the square of 70, from the given number, and the remainder is 429, which must be equal to $(2a+b)b$. Divide this remainder by $2a$, that is by 140, and the quotient is 3, which is the value of b. Completing the divisor, we have $2a+b=143$; whence $(2a+b)b$, that is 143×3, or 429, is the quantity to be subtracted; and as there is now no remainder, we conclude that 70+3, or 73, is the required square root.

For the sake of brevity, the ciphers may be omitted, provided we retain the proper local values of the figures.

If the root consists of three places of figures, let a represent the hundreds, and b the tens; then, having obtained a and b as before, let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units.

Required the square root of 568516.

```
        56,85,16 ( 754
        49
 145 )   785
         725
1504 )    6016
          6016
```

Having found 75, the square root of the greatest square number contained in the first two periods, we bring down the last period, and have 6016 for a new dividend. We then take $2a$, or 150, for a partial divisor, whence we obtain $b=4$ for the last figure of the root. The entire root is therefore 754.

203. Hence, for the extraction of the square root of numbers, we derive the following

RULE.

1*st. Separate the given number into periods of two figures each, beginning from the units' place.*

2*d. Find the greatest number whose square is contained in the left-hand period; this is the first figure of the required root. Subtract its square from the first period, and to the remainder bring down the second period for a dividend.*

3*d. Double the root already found for a divisor, and find how many times it is contained in the dividend, exclusive of its right-hand figure; annex the result both to the root and the divisor.*

4*th. Multiply the divisor thus increased by the last figure of the root, subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

5*th. Double the whole root now found for a new divisor, and proceed as before, continuing the operation until all the periods are brought down.*

In applying the preceding rule, it may happen that the product of the complete divisor by the last figure of the root is greater than the dividend. This indicates that the last figure of the root was taken too large, and this happens because the divisor is at first incomplete—that is, is *too small.* In such a case, we must diminish the last figure of the root by unity until we obtain a product which is not greater than the dividend.

EXAMPLES.

1. What is the square root of 294849? *Ans.* 543.
2. What is the square root of 840889?
3. What is the square root of 1142761?
4. What is the square root of 32239684?
5. What is the square root of 72777961?
6. What is the square root of 3518743761?

204. *Square Root of Fractions.*—We have seen that the root of a fraction is equal to the root of its numerator divided by the root of its denominator. Hence the square root of $\frac{64}{169}$ is $\frac{8}{13}$.

The number 5.29 may be written $\frac{529}{100}$, and its square root is $\frac{23}{10}$, or 2.3. So, also, 18.6624 may be written $\frac{186624}{10000}$, and its square root is $\frac{432}{100}$, or 4.32. That is, the square root of a decimal fraction, or of a whole number followed by a decimal fraction, may be found in the same manner as that of a whole number, if we divide it into periods *commencing with the decimal point.*

In the extraction of the square root of an integer, if there is still a remainder after we have obtained the units' figure of the root, it indicates that the proposed number has not an exact square root. We may, if we please, proceed with the approximation to any desired extent by supposing a decimal point at the end of the proposed number, and annexing any even number of ciphers, and continuing the operation. We thus obtain a decimal part to be added to the integral part already found.

So, also, if a decimal number has no exact square root, we may annex ciphers and proceed with the approximation to any desired extent.

EXAMPLES.

1. What is the square root of $\frac{1089}{2809}$? *Ans.* $\frac{33}{53}$.
2. What is the square root of $\frac{18769}{170569}$?
3. What is the square root of $\frac{1}{67551961}$?
4. What is the square root of 9.878449?
5. What is the square root of 58.614336?
6. What is the square root of .558009?
7. What is the square root of .03478225?

Find the square roots of the following numbers to five decimal places.

8. Of 2. *Ans.* 1.41421.
9. Of 10. *Ans.* 3.16227.
10. Of 9.1.
11. Of $4\frac{3}{8}$.
12. Of $\frac{1}{13}$.
13. Of $\frac{5}{17}$.

Cube Root of a Polynomial.

205. We already know that the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$. If, then, the *cube* were given, and we were required to find its *root*, it might be done by the following method.

When the terms are arranged according to the powers of one letter, a, we at once know, from the first term, a^3, that a must be one term of the root. If, then, we subtract its cube from the proposed polynomial, we obtain the remainder $3a^2b+3ab^2+b^3$, which must furnish the second term of the root.

Now this remainder may be put under the form

$$(3a^2+3ab+b^2)b;$$

whence it appears that we shall find the second term of the root if we divide the remainder by $3a^2+3ab+b^2$. But, as this second term is supposed to be unknown, the divisor can not be completed. Nevertheless, we know the first term, $3a^2$, that is thrice the square of the first term already found, and by means of this we can find the other part, b; viz., by dividing the first term of the remainder by $3a^2$. We then complete the divisor by adding to it $3ab+b^2$. If this complete divisor be multiplied by b, it will give the last three terms of the power.

Let it be required to find the cube root of $8a^3+36a^2b+54ab^2+27b^3$.

$$\begin{array}{r|l} & 8a^3+36a^2b+54ab^2+27b^3\ (2a+3b \\ & \underline{8a^3} \\ 12a^2+18ab+9b^2\ ) & 36a^2b+54ab^2+27b^3 \\ & \underline{36a^2b+54ab^2+27b^3} \end{array}$$

Having found the first term of the root, $2a$, and subtracted its cube, we divide the first term of the remainder, $36a^2b$, by three times the square of $2a$, that is $12a^2$, and we obtain $3b$ for the second term of the root. We then complete the divisor by adding to it three times the product of the two terms of the root, which is $18ab$, together with the square of the last term $3b$, which is $9b^2$. Multiplying then the complete divisor by $3b$, and subtracting the product from the last remainder, nothing is left. Hence the required cube root is $2a+3b$.

This result may be easily verified by multiplication.

206. If the root contains *three* terms, as $a+b+c$, we may put $a+b=m$. Then

$$(a+b+c)^3=(m+c)^3=m^3+3m^2c+3mc^2+c^3.$$

If we proceed as in the last example, we shall find $a+b$, and we subtract its cube from the given polynomial. There will then remain $3m^2c+3mc^2+c^3$, which may be written

$$(3m^2+3mc+c^2)c.$$

We perceive that $3m^2$ will be the new trial divisor to obtain c. We then complete the divisor by adding to it $3mc+c^2$.

Let it be required to find the cube root of $8a^6-36ba^5+66b^2a^4-63b^3a^3+33b^4a^2-9b^5a+b^6$.

$$8a^6-36ba^5+66b^2a^4-63b^3a^3+33b^4a^2-9b^5a+b^6\,(2a^2-3ba+b^2$$
$$8a^6$$
$$12a^4-18ba^3+9b^2a^2\,)\,-36ba^5+66b^2a^4-63b^3a^3$$
$$-36ba^5+54b^2a^4-27b^3a^3$$
$$\begin{matrix}12a^4-36ba^3+27b^2a^2 \\ +\ 6b^2a^2-9b^3a+b^4\end{matrix}\Bigg)\,12b^2a^4-36b^3a^3+33b^4a^2-9b^5a+b^6$$
$$12b^2a^4-36b^3a^3+33b^4a^2-9b^5a+b^6$$

The first term of the root is $2a^2$, and subtracting its cube, the first term of the remainder is $-36ba^5$, which, divided by 3 times the square of $2a^2$, gives $-3ba$ for the second term of the root. Complete the divisor as in the last example, and multiply it by $-3ba$. Subtracting the product from the last remainder, the first term of the second remainder is $12b^2a^4$.

To form the new trial divisor, we take three times the square of the part of the root already found, viz., $2a^2-3ba$. Divide the first term of the remainder by $12a^4$, and we obtain b^2 for the last term of the root. We now complete the divisor by adding to it three times the product of the third term by the sum of the first two terms, and also the square of the last term. Multiplying the divisor thus completed by b^2, we find the product equal to the last remainder. Hence the required cube root is $2a^2-3ba+b^2$.

207. Hence, for extracting the cube root of a polynomial, we derive the following

RULE.

1*st. Arrange the terms according to the powers of some one letter; take the cube root of the first term, and subtract the cube from the given polynomial.*

2*d. Divide the first term of the remainder by three times the square of the root already found; the quotient will be the second term of the root.*

3*d. Complete the divisor by adding to it three times the product of the two terms of the root and the square of the second term.*

4*th. Multiply the divisor thus increased by the last term of the root, and subtract the product from the last remainder.*

5*th. Take three times the square of the part of the root already found for a new trial divisor, and proceed by division to find another term of the root.*

6*th. Complete the divisor by adding to it three times the product of the last term by the sum of the first two terms, and also the square of the last term, with which proceed as before till the entire root has been obtained.*

We may dispense with forming the complete divisor according to the rule if each time that we find a new term of the root we raise the entire root already found to the third power, and subtract the cube from the given polynomial.

EXAMPLES.

1. What is the cube root of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$? *Ans.* a^2-2a+1.

2. What is the cube root of $6x^5-40x^3+x^6+96x-64$? *Ans.*

3. What is the cube root of $18x^4+36x^2+24x+8+32x^3+x^6+6x^5$? *Ans.*

4. What is the cube root of $3b^5+b^6-5b^3-1+3b$? *Ans.*

5. What is the cube root of $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$?

6. What is the cube root of $8x^6+48ax^5+60a^2x^4-80a^3x^3-90a^4x^2+108a^5x-27a^6$?

7. What is the cube root of $8x^6-36ax^5+102a^2x^4-171a^3x^3+204a^4x^2-144a^5x+64a^6$?

Cube Root of Numbers.

208. The preceding rule is applicable to the extraction of the cube root of numbers; but a difficulty in applying it arises from the fact that the terms of the powers are all blended together in the given number. They may, however, be separated by attending to the following principles:

1*st. For every three figures of the cube there will be one figure in the root, and also one for any additional figure or figures.* Thus,

the cube of	1 is	1		the cube of	1 is	1
"	9 "	729		"	10 "	1,000
"	99 "	970,299		"	100 "	1,000,000
"	999 "	997,002,999		"	1000 "	1,000,000,000

Hence we see that the cube root of a number consisting of from one to three figures will contain *one* figure; the cube root of a number consisting of from four to six figures will contain *two* figures; of a number from seven to nine figures will contain *three* figures, and so on.

Hence, if we divide the number into periods of three figures, commencing at units' place, the number of periods will indicate the number of figures in the cube root.

209. 2*d. The first figure of the root will be the cube root of the greatest cube number contained in the first period on the left.*

For the cube of tens can give no significant figure in the first right-hand period; the cube of hundreds can give no figure in the first two periods on the right; and the cube of the highest figure in the root can give no figures except in the first period on the left.

Let it be required to extract the cube root of 438976.

This number contains two periods, indicating that there will be two places in the root. Let a be the value of the figure in the tens' place, and b of that in the units' place. Then a must be the greatest multiple of 10 which has its cube less than 438000; that is, a must be 70. Subtract the cube of 70 from the given number, and the remainder is 95976. This remainder corresponds to $3a^2b+3ab^2+b^3$, which may be written

	438,976(70+6, the root.
	343,000
$70^2\times3=14700$	
$70\times6\times3=\ 1260$	95976
$6^2=\ \ 36$	95976
complete divisor, 15996	

$$(3a^2+3ab+b^2)b.$$

Divide this remainder by $3a^2$, that is, by 14700, and the quotient is 6, which is the value of b. Complete the divisor by adding to it $3ab$, or 1260, and b^2, or 36. The complete divisor is thus found to be 15996, which, multiplied by 6, gives 95976. Subtracting, the remainder is zero, and we conclude that 70+6, or 76, is the required cube root.

For the sake of brevity the ciphers may be omitted, provided we retain the proper local values of the figures.

If the root consists of more than two places of figures, the method will be substantially the same.

Let it be required to extract the cube root of 279,726,264.

	279,726,264(654
	216
108	
90	63726
25	
11725	58625
12675	
780	5101 264
16	
1275316	5101 264

Having found 65, the cube root of the greatest cube contained in the first two periods, we bring down the last period, and have 5101264 for a new dividend. We then take three times the square of the root already found, or 12675, for a partial divisor, whence we obtain 4 for the last figure of the root. We then complete the divisor by adding to it three times the product of 4 by 65, and the square of 4, regard being paid to the proper local values of the figures. The complete divisor is thus found to be 1275316, which, multiplied by 4, gives 5101264. Hence 654 is the required cube root.

210. Hence, for the extraction of the cube root of numbers, we derive the following

RULE.

1*st. Separate the given number into periods of three figures each, beginning at the units' place.*

2*d. Find the greatest cube contained in the left-hand period; its cube root is the first figure of the required root. Subtract the cube from the first period, and to the remainder bring down the second period for a dividend.*

3*d. Take three hundred times the square of the root already found for a trial divisor; find how many times it is contained in the dividend, and write the quotient for the second figure of the root.*

4*th. Complete the divisor by adding to it thirty times the product of the two figures of the root, and the square of the second figure.*

5*th. Multiply the divisor thus increased by the last figure of the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

6*th. Take three hundred times the square of the whole root now found for a new trial divisor, and by division obtain another figure of the root.*

7*th. Complete the divisor by adding to it thirty times the product of the last figure by the former figures, and also the square of the last figure, with which proceed as before, continuing the operation until all the periods are brought down.*

It will be observed that three times the square of the tens, when their local value is regarded, is the same as three hundred times the square of this digit, *not* regarding its local value.

In applying the preceding rule, it may happen that the product of the complete divisor by the last figure of the root is greater than the dividend. This indicates that the last figure of the root was taken too large, and this happens because the divisor is at first incomplete, that is, *too small.* In such a case we must diminish the last figure of the root by unity, until we obtain a product which is not greater than the dividend.

EXAMPLES.

1. Find the cube root of 163667323. *Ans.* 547.

2. Find the cube root of 39651821. *Ans.* 341.
3. Find the cube root of 4019679. *Ans.* 159.
4. Find the cube root of 12895213625.
5. Find the cube root of 183056926752.
6. Find the cube root of 759299343867.

211. *Cube Root of Fractions.*—The cube root of a fraction is equal to the root of its numerator divided by the root of its denominator. Hence the cube root of $\frac{343}{1728}$ is $\frac{7}{12}$.

The number 12.167 may be written $\frac{12167}{1000}$, and its cube root is $\frac{23}{10}$, or 2.3. That is, the cube root of a decimal fraction, or of a whole number followed by a decimal fraction, may be found in the same manner as that of a whole number, if we divide it into periods *commencing with the decimal point.*

In the extraction of the cube root of an integer, if there is still a remainder after we have obtained the units' figure of the root, it indicates that the proposed number has not an exact cube root. We may, if we please, proceed with the approximation to any desired extent, by supposing a decimal point at the end of the proposed number, and annexing any number of periods of three ciphers each, and continuing the operation. We thus obtain a decimal part to be added to the integral part already found.

So, also, if a decimal number has no exact cube root, we may annex ciphers, and proceed with the approximation to any desired extent.

EXAMPLES.

1. Find the cube root of $\frac{6859}{15625}$. *Ans.* $\frac{19}{25}$.
2. Find the cube root of $14\frac{111}{343}$. *Ans.* $2\frac{3}{7}$.
3. Find the cube root of 13.312053.
4. Find the cube root of 1892.819053.
5. Find the cube root of .001879080904.

Find the cube roots of the following numbers to 5 decimal places:

6. 15.25.	*Ans.* 2.47984.	10. 11.
7. 3.7.	*Ans.* 1.54668.	11. $\frac{1}{3}$.
8. 100.1.	*Ans.* 4.64314.	12. $\frac{3}{4}$.
9. 4.	*Ans.* 1.58740.	

CHAPTER XIII.

RADICAL QUANTITIES.

212. A *radical* quantity is an indicated root of a quantity: as $\sqrt{a}$, $\sqrt[3]{a}$, etc. Radical quantities may be either surd or rational.

Radical quantities are divided into *degrees*, the degree being denoted by the index of the root. Thus, $\sqrt{3}$ is a radical of the second degree; $\sqrt[3]{5}$ is a radical of the third degree, etc.

213. The *coefficient* of a radical is the number or letter prefixed to it, showing how often the radical is to be taken. Thus, in the expression $2\sqrt{a}$, 2 is the coefficient of the radical.

Similar radicals are those which have the same index and the same quantity under the radical sign. Thus, $3\sqrt{a}$ and $5\sqrt{a}$ are similar radicals. Also $7\sqrt[3]{b}$ and $10\sqrt[3]{b}$ are similar radicals.

214. *Use of fractional Exponents.*—We have seen, Art. 196, that in order to extract any root of a monomial, we must divide the exponent of each literal factor by the index of the required root. Thus the square root of a^4 is a^2, and in the same manner the square root of a^3 may be written $a^{\frac{3}{2}}$, that of a^5 will be $a^{\frac{5}{2}}$, and that of a, or a^1, is $a^{\frac{1}{2}}$. Whence we see that

$$a^{\frac{1}{2}} \text{ is equivalent to } \sqrt{a},$$
$$a^{\frac{3}{2}} \quad \text{“} \quad \sqrt{a^3},$$
$$a^{\frac{n}{2}} \quad \text{“} \quad \sqrt{a^n}, \text{ etc.}$$

So, also, the cube root of a^2 may be written $a^{\frac{2}{3}}$; the cube root of a^4 is $a^{\frac{4}{3}}$; and the cube root of a, or a^1, is $a^{\frac{1}{3}}$. Whence we see that

$$a^{\frac{1}{3}} \text{ is equivalent to } \sqrt[3]{a},$$
$$a^{\frac{4}{3}} \quad \text{"} \quad \sqrt[3]{a^4},$$
$$a^{\frac{n}{3}} \quad \text{"} \quad \sqrt[3]{a^n}, \text{ etc.}$$

In the same manner, $a^{\frac{1}{4}}$ is equivalent to $\sqrt[4]{a}$,
$$a^{\frac{1}{5}} \quad \text{"} \quad \sqrt[5]{a},$$
$$a^{\frac{5}{4}} \quad \text{"} \quad \sqrt[4]{a^5},$$
$$a^{\frac{m}{n}} \quad \text{"} \quad \sqrt[n]{a^m}.$$

That is, *the numerator of a fractional exponent denotes the power, and the denominator the root to be extracted.*

Let it be required to extract the cube root of $\frac{1}{a^4}$. This quantity, Art. 187, is equivalent to a^{-4}. Now, to extract the cube root of a^{-4}, we must divide its exponent by 3, which gives us $a^{-\frac{4}{3}}$. But the cube root of $\frac{1}{a^4}$ may also be represented by $\frac{1}{a^{\frac{4}{3}}}$.

Hence $\frac{1}{a^{\frac{4}{3}}}$ is equivalent to $a^{-\frac{4}{3}}$.

So, also, $\frac{1}{a^{\frac{1}{2}}}$ is equivalent to $a^{-\frac{1}{2}}$,
$$\frac{1}{a^{\frac{1}{n}}} \quad \text{"} \quad a^{-\frac{1}{n}},$$
$$\frac{1}{a^{\frac{m}{n}}} \quad \text{"} \quad a^{-\frac{m}{n}}.$$

Thus we see that the principle of Art. 77, that a factor may be transferred from the numerator to the denominator of a fraction, or from the denominator to the numerator by changing the sign of its exponent, is *applicable also to fractional exponents.*

We may therefore entirely reject the radical signs hitherto employed, and substitute for them fractional exponents, and many of the difficulties which occur in the reduction of radical quantities are thus made to disappear.

To reduce a Radical to its simplest Form.

215. A radical is in its simplest form when it has under the radical sign no factor which is a perfect power corresponding to the degree of the radical.

Radical quantities may frequently be simplified by the application of the following principle: *the nth root of the product of two or more factors is equal to the product of the nth roots of those factors;* or, in algebraic language,

$$\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}.$$

For each of these expressions, raised to the nth power, will give the same quantity.

Thus, the nth power of $\sqrt[n]{ab}$ is ab.

And the nth power of $\sqrt[n]{a}\times\sqrt[n]{b}$ is $(\sqrt[n]{a})^n\times(\sqrt[n]{b})^n$, or ab.

Hence, since the *same powers* of the quantities $\sqrt[n]{ab}$ and $\sqrt[n]{a}\times\sqrt[n]{b}$ are equal, the quantities themselves must be equal.

Let it be required to reduce $\sqrt{48a^3x^2}$ to its simplest form.

This expression may be put under the form $\sqrt{16a^2x^2}\times\sqrt{3a}$.

But $\sqrt{16a^2x^2}$ is equal to $4ax$.

Hence $\sqrt{48a^3x^2}=4ax\sqrt{3a}$.

Hence, to reduce a radical to its simplest form, we have the following

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the greatest perfect power corresponding in degree to the radical. Extract the required root of this factor, and prefix it to the other factor, which must be left under the sign.

EXAMPLES.

Reduce the following radicals to their simplest forms.

1. $\sqrt{125a^3}$. *Ans.* $5a\sqrt{5a}$.
2. $\sqrt{98ab^4}$. *Ans.* $7b^2\sqrt{2a}$.
3. $\sqrt{294ab^2}$.
4. $7\sqrt{80abc^3}$.
5. $\sqrt{98a^2x^6y^2}$.

6. $\sqrt{45a^2b^3c^2x}$.

7. $\sqrt{864a^2b^5x^{11}}$.

8. $\sqrt[3]{56a^5b^6}$. *Ans.* $2ab^2\sqrt[3]{7a^2}$.

9. $\sqrt[3]{54a^4b^3c^2}$. *Ans.* $3ab\sqrt[3]{2ac^2}$.

10. $\sqrt[4]{48a^5b^8c^6}$. *Ans.* $2ab^2c\sqrt[4]{3ac^2}$.

11. $\sqrt[6]{192a^7bx^{12}}$.

12. $9\sqrt[3]{81a^2b^3x}$.

13. $\sqrt{a^3-a^2x}$.

216. When the quantity under the radical sign is a *fraction*, it is often convenient to multiply both its terms by such a quantity as will make the denominator a perfect power of the degree indicated. Then, after simplifying, the factor remaining under the radical sign will be entire.

14. $3\sqrt{\frac{2}{5}}$. *Ans.* $3\sqrt{\frac{2}{5}}=3\sqrt{\frac{10}{25}}=\frac{3}{5}\sqrt{10}$.

15. $\frac{5}{2}\sqrt{\frac{1}{2}}+6\sqrt{3\frac{1}{2}}$. *Ans.* $\frac{5}{4}\sqrt{2}+3\sqrt{14}$.

16. $\sqrt{8\frac{1}{3}}+2\sqrt{\frac{5}{3}}$. *Ans.* $\frac{5}{3}\sqrt{3}+\frac{2}{3}\sqrt{15}$.

17. $\sqrt{1\frac{4}{5}}+\frac{1}{2}\sqrt{\frac{49}{5}}$. *Ans.* $\frac{3}{5}\sqrt{5}+\frac{7}{10}\sqrt{5}$.

18. $\frac{3}{4}\sqrt{\frac{5}{6}}+3\sqrt{\frac{2}{7}}$. *Ans.* $\frac{1}{8}\sqrt{30}+\frac{3}{7}\sqrt{14}$.

19. $15\sqrt{\frac{5}{12}}+5\sqrt{\frac{5}{18}}$. *Ans.* $\frac{5}{2}\sqrt{15}+\frac{5}{6}\sqrt{10}$.

20. $a\sqrt{\dfrac{b}{a}}+ab\sqrt{\dfrac{1}{ab}}$. *Ans.* $2\sqrt{ab}$.

21. $(a-b)\sqrt{\dfrac{a+b}{a-b}}$. *Ans.* $\sqrt{a^2-b^2}$.

22. $\dfrac{a^3b^2}{c^2}\sqrt{\dfrac{c^5}{a^5b^5}}$. *Ans.* $\dfrac{1}{b}\sqrt{abc}$.

23. $2\sqrt[3]{\frac{1}{2}}+3\sqrt[3]{\frac{1}{3}}$. *Ans.* $\sqrt[3]{4}+\sqrt[3]{9}$.

24. $2\sqrt[3]{2\frac{2}{3}}+7\sqrt[3]{7\frac{19}{32}}$. *Ans.* $\frac{4}{3}\sqrt[3]{9}+\frac{21}{4}\sqrt[3]{18}$.

217. The following principle can frequently be employed in simplifying radicals:

The mnth root of any quantity is equal to the mth root of the nth root of that quantity. That is,

$$\sqrt[mn]{a}=\sqrt[m]{\sqrt[n]{a}}.$$

For, if we raise each expression to the mth power, it becomes $\sqrt[n]{a}$.

Thus, the fourth root = the square root of the square root.
the sixth root = the square root of the cube root, or the cube root of the square root.
the eighth root = the square root of the fourth root, or the fourth root of the square root.
the ninth root = the cube root of the cube root.

Hence, *when the index of a root is the product of two or more factors, we may obtain the root required by extracting in succession the roots denoted by those factors.*

Ex. 1. Let it be required to extract the sixth root of 64.

The square root of 64 is 8, and the cube root of 8 is 2. Hence the sixth root of 64 is 2.

Ex. 2. Extract the eighth root of 256. *Ans.* 2.

Ex. 3. Find the fourth root of 1874161. *Ans.* 37.

Ex. 4. Find the sixth root of 148035889. *Ans.* 23.

Ex. 5. Find the ninth root of 387420489. *Ans.* 9.

Ex. 6. Find the eighth root of 2562890625. *Ans.* 15.

218. When the index of a root is the product of two or more factors, and one of the roots can be extracted, while the other can not, *a radical may be simplified by extracting one of the roots.*

Thus, $\sqrt[4]{9}=\sqrt{3}$.

Reduce the following radicals to their simplest forms:

Ex. 1. $\sqrt[6]{4a^2}$. *Ans.* $\sqrt[3]{2a}$.

Ex. 2. $\sqrt[4]{36a^2b^2}$.

Ex. 3. $\sqrt[mn]{a^n}$.

Ex. 4. $\sqrt[8]{25a^4b^2c^6}$.

Ex. 5. $\sqrt[6]{\frac{25a^2}{64b^2}}$.

Ex. 6. $\sqrt[9]{\frac{a^{17}b^3c^5}{m^8n^5}}$. *Ans.* $\frac{a^5bc}{m^2n}\sqrt[3]{\frac{a^2c^2}{m^2n^2}}$.

To introduce a Factor under the Radical Sign.

219. The square root of the square of a is obviously a, and the cube root of the cube of a is a, etc. That is,

$$a=\sqrt{a^2}=\sqrt[3]{a^3}=\sqrt[4]{a^4}, \text{ etc.}$$

Whence, also, $$a\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b}.$$

Hence, to introduce a factor under the radical sign, we have the following

RULE.

Raise the factor to a power denoted by the index of the required root, and write it as a factor under the radical sign.

EXAMPLES.

1. Reduce ax^2 to a radical of the second degree.
Ans. $\sqrt{a^2x^4}$.
2. Reduce $2a^2bx$ to a radical of the third degree.
Ans. $\sqrt[3]{8a^6b^3x^3}$.
3. Reduce $5+b$ to a radical of the second degree.
Ans. $\sqrt{25+10b+b^2}$.

Transform the following radicals by introducing the coefficients as factors under the radical sign:

4. $6\sqrt{3\frac{1}{2}}$. *Ans.* $\sqrt{126}$.
5. $4\sqrt{\frac{1}{8}}+3\frac{1}{2}\sqrt{8}$. *Ans.* $\sqrt{2}+\sqrt{98}$.
6. $a\sqrt{\dfrac{b}{a}}+ab\sqrt{\dfrac{1}{ab}}$. *Ans.* $2\sqrt{ab}$.
7. $(a+b)\sqrt{\dfrac{ab}{a^2+2ab+b^2}}$. *Ans.* $\sqrt{ab}$.
8. $2\sqrt[3]{2}+7\sqrt[3]{5}$. *Ans.* $\sqrt[3]{16}+\sqrt[3]{1715}$.
9. $7\frac{1}{9}\sqrt[3]{7\frac{19}{32}}$. *Ans.* $\sqrt[3]{\frac{8192}{3}}$.

To change the Index of a Radical.

220. From Art. 219, it follows that

$$\sqrt[2]{a}=\sqrt[4]{a^2}=\sqrt[6]{a^3}=\sqrt[8]{a^4}, \text{ etc.};$$
$$\sqrt[3]{a}=\sqrt[6]{a^2}=\sqrt[9]{a^3}=\sqrt[12]{a^4}, \text{ etc.};$$
$$\sqrt[n]{a}=\sqrt[2n]{a^2}=\sqrt[3n]{a^3}=\sqrt[4n]{a^4}, \text{ etc.}$$

Hence we see that *the index of any radical may be multiplied by any number, provided we raise the quantity under the radical sign to a power whose exponent is the same number;* or *the index of any radical may be divided by any number, provided we extract that root of the quantity under the radical sign whose index is the same number.*

If, instead of the radical sign, we employ fractional exponents, we shall have

$$a^{\frac{1}{2}}=a^{\frac{2}{4}}=a^{\frac{3}{6}}=a^{\frac{4}{8}}, \text{ etc.}$$

$$a^{\frac{1}{3}}=a^{\frac{2}{6}}=a^{\frac{3}{9}}=a^{\frac{4}{12}}, \text{ etc.}$$

Hence we see that *we may multiply or divide both terms of a fractional exponent by the same number, without changing the value of the expression.*

EXAMPLES.

Verify the following equations:

1. $\sqrt{2}+\sqrt[3]{3}+\sqrt[4]{4}=\sqrt[4]{4}+\sqrt[6]{9}+\sqrt[8]{16}$.
2. $\sqrt[4]{a^6b^8}+\sqrt[4]{\dfrac{x^4}{4}}=\sqrt{a^3b^4}+\sqrt{\tfrac{1}{2}x^2}$.
3. $2\sqrt[4]{a^2-2ab+b^2}=2\sqrt{a-b}$.
4. $2\sqrt[3]{a-b}=2\sqrt[9]{a^3-3a^2b+3ab^2-b^3}$.
5. $3\sqrt[6]{8x^3-12x^2y+6xy^2-y^3}=3\sqrt{2x-y}$.

To reduce Radicals to a Common Index.

221. Let it be required to reduce $\sqrt{a}$ and $\sqrt[3]{a}$ to equivalent radicals having a common index. Substituting for the radical signs fractional exponents, the given quantities are

$$a^{\frac{1}{2}} \text{ and } a^{\frac{1}{3}}.$$

Reducing the exponents to a common denominator, the expressions are

$$a^{\frac{3}{6}} \text{ and } a^{\frac{2}{6}},$$

or

$$\sqrt[6]{a^3} \text{ and } \sqrt[6]{a^2},$$

which are of the same value as the given quantities, and have a common index 6. Hence we derive the following

RULE.

Reduce the fractional exponents to a common denominator, raise each quantity to the power denoted by the numerator of its new exponent, and take the root denoted by the common denominator.

EXAMPLES.

1. Reduce $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, and $a^{\frac{1}{4}}$ to a common index.

   Ans. $a^{\frac{6}{12}}$, $a^{\frac{4}{12}}$, and $a^{\frac{3}{12}}$.

2. Reduce $a^{\frac{1}{2}}$, a^{2}, and $b^{\frac{2}{3}}$ to a common index.

   Ans. $a^{\frac{3}{6}}$, $a^{\frac{12}{6}}$, and $b^{\frac{4}{6}}$.

3. Reduce $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, and $5^{\frac{1}{4}}$ to a common index.

   Ans. $\sqrt[12]{64}$, $\sqrt[12]{81}$, and $\sqrt[12]{125}$.

4. Reduce $3^{\frac{1}{2}}$, $2^{\frac{2}{3}}$, and $2^{\frac{3}{4}}$ to a common index.

   Ans. $\sqrt[12]{729}$, $\sqrt[12]{256}$, and $\sqrt[12]{512}$.

5. Reduce $a^{\frac{1}{2}}$, $a^{\frac{1}{m}}$, and $a^{\frac{1}{n}}$ to a common index.

   Ans. $a^{\frac{mn}{2mn}}$, $a^{\frac{2n}{2mn}}$, and $a^{\frac{2m}{2mn}}$.

6. Reduce $\sqrt[3]{3}$, $\sqrt{5}$, and $\sqrt[6]{7}$ to a common index.

7. Reduce $\sqrt{2ab}$, $\sqrt[3]{3ab^2}$, and $\sqrt[6]{5ab^3}$ to a common index.

8. Reduce $\sqrt{a+b}$, $\sqrt[4]{a-b}$, and $\sqrt[5]{a^2-b^2}$ to a common index.

To add Radical Quantities together.

222. When the radical quantities are *similar*, the common radical part may be regarded as the *unit*, and the coefficient shows how many times this unit is repeated. The sum of the coefficients of the given radicals will then denote how many times this unit is to be repeated in the required sum.

If the radicals are *not similar* they can not be added, because they have no common unit. In such a case, the addition can only be indicated by the algebraic sign. Radicals which are apparently dissimilar may become similar when reduced to their simplest forms. Hence we have the following

RULE.

Reduce each radical to its simplest form. If the resulting radicals are similar, add their coefficients, and to their sum annex the common radical. If they are dissimilar, connect them by the sign of addition.

EXAMPLES.

1. Find the sum of $\sqrt{27}$, $\sqrt{48}$, and $\sqrt{75}$. *Ans.* $12\sqrt{3}$.

2. Find the sum of $4\sqrt{147}$, $3\sqrt{75}$, and $\sqrt{192}$. *Ans.* $51\sqrt{3}$.

3. Find the sum of $\sqrt{72}$, $\sqrt{128}$, and $\sqrt{162}$. *Ans.* $23\sqrt{2}$.

4. Find the sum of $\sqrt{180}$, $\sqrt{405}$, and $\sqrt{320}$. *Ans.* $23\sqrt{5}$.

5. Find the sum of $3\sqrt{\frac{2}{5}}$, $2\sqrt{\frac{1}{10}}$, and $4\sqrt{\frac{1}{40}}$. *Ans.* $\sqrt{10}$.

6. Find the sum of $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{256}$. *Ans.* $12\sqrt[3]{4}$.

7. Find the sum of $\sqrt[3]{40}$, $\sqrt[3]{135}$, and $\sqrt[3]{320}$. *Ans.* $9\sqrt[3]{5}$.

8. Find the sum of $2\sqrt{\frac{5}{3}}$, $\sqrt{60}$, $\sqrt{15}$, and $\sqrt{\frac{3}{5}}$. *Ans.* $\frac{58}{15}\sqrt{15}$.

9. Find the sum of $\sqrt{45c^3}$, $\sqrt{80c^3}$, and $\sqrt{5a^2c}$. *Ans.* $(a+7c)\sqrt{5c}$.

10. Find the sum of $\sqrt{18a^5b^3}+\sqrt{50a^3b^3}$. *Ans.* $(3a^2b+5ab)\sqrt{2ab}$.

11. Find the sum of $\sqrt{\frac{a^4c}{b^3}}$, $\sqrt{\frac{a^2c^3}{bd^2}}$, and $\sqrt{\frac{a^2cd^2}{bm^2}}$.

    Ans. $\left(\frac{a^2}{b}+\frac{ac}{d}+\frac{ad}{m}\right)\sqrt{\frac{c}{b}}$.

12. Find the sum of $\sqrt{4a^3b}$, $\sqrt{25ab^3}$, and $5b\sqrt{ab}$. *Ans.* $(2a+6b)\sqrt{ab}$.

To find the Difference of Radical Quantities.

223. When the radicals are similar, it is evident that the subtraction may be performed in the same manner as addition, except that the signs in the subtrahend are to be changed. Hence we have the following

RULE.

Reduce each radical to its simplest form. If the resulting radicals are similar, find the difference of the coefficients, and to the result annex the common radical part. If they are dissimilar, the subtraction can only be indicated.

EXAMPLES.

1. From $\sqrt{448}$ take $\sqrt{112}$. *Ans.* $4\sqrt{7}$.
2. From $5\sqrt{20}$ take $3\sqrt{45}$. *Ans.* $\sqrt{5}$.
3. From $2\sqrt{50}$ take $\sqrt{18}$. *Ans.* $7\sqrt{2}$.
4. From $\sqrt{80a^4x}$ take $\sqrt{20a^2x^3}$.
5. From $2\sqrt{72a^2}$ take $\sqrt{162a^2}$.
6. From $\sqrt[3]{192}$ take $\sqrt[3]{24}$. *Ans.* $2\sqrt[3]{3}$.
7. From $2\sqrt[3]{320}$ take $3\sqrt[3]{40}$.
8. From $\sqrt[3]{\frac{27a^5x}{2b}}$ take $\sqrt[3]{\frac{a^2x}{2b}}$. *Ans.* $(3a-1)\sqrt[3]{\frac{a^2x}{2b}}$.

To multiply Radical Quantities together.

224. We have found, Art. 215, that $\sqrt[n]{a}$ multiplied by $\sqrt[n]{b}$ is equal to $\sqrt[n]{ab}$.

Hence, $\sqrt{2}\times\sqrt{3}=\sqrt{6}$.

If the radicals have coefficients, the product of the coefficients may be taken separately.

Thus, $a\sqrt{x}\times b\sqrt{y}=a\times b\times\sqrt{x}\times\sqrt{y}=ab\sqrt{xy}$;

also, $3\sqrt{8}\times 5\sqrt{2}=15\sqrt{16}=60$.

If the radicals have not a common index, they must first be reduced to a common index. Hence we have the following

RULE.

If necessary, reduce the given radicals to a common index. Multiply the coefficients together for a new coefficient; also multiply the quantities under the radical signs together, and place this product under the common radical sign. Then reduce the result to its simplest form.

EXAMPLES.

Find the value of the following expressions:

1. $3\sqrt{8}\times2\sqrt{6}$. *Ans.* $24\sqrt{3}$.
2. $5\sqrt{8}\times3\sqrt{5}$. *Ans.* $30\sqrt{10}$.
3. $\sqrt{2}\times\sqrt[3]{3}$. *Ans.* $\sqrt[6]{72}$.
4. $5\sqrt{3}\times7\sqrt{\frac{8}{3}}\times\sqrt{2}$. *Ans.* 140.
5. $c\sqrt{a}\times d\sqrt{a}$. *Ans.* acd.
6. $7\sqrt[3]{18}\times5\sqrt[3]{4}$. *Ans.* $70\sqrt[3]{9}$.
7. $\frac{1}{4}\sqrt[3]{6}\times\frac{2}{15}\sqrt[3]{17}$.
8. $\frac{1}{2}\sqrt[3]{18}\times5\sqrt[3]{20}$.
9. $\sqrt[3]{4}\times7\sqrt[3]{6}\times\frac{1}{2}\sqrt[3]{5}$. *Ans.* $7\sqrt[3]{15}$.

225. We have seen, *Art.* 58, that *the exponent of any letter in a product is equal to the sum of the exponents of this letter in the multiplicand and multiplier.* That is, $a^m\times a^n=a^{m+n}$, where m and n are supposed to be *positive whole numbers.*

When one or both of the exponents are *negative*, we must take the *algebraic sum* of the exponents. For, suppose n is negative. Then

$$a^m\times a^{-n}=a^m\times\frac{1}{a^n}, \text{ by Art. 76, } =\frac{a^m}{a^n}=a^{m-n}.$$

The same relation holds true when m and n are *fractional;* that is,

$$a^{\frac{p}{q}}\times a^{\frac{r}{s}}=a^{\frac{p}{q}+\frac{r}{s}}.$$

For $a^{\frac{p}{q}}\times a^{\frac{r}{s}}=\sqrt[q]{a^p}\times\sqrt[s]{a^r}$, Art. 214, $=\sqrt[qs]{a^{ps}}\times\sqrt[qs]{a^{qr}}$, Art. 220,

$$=\sqrt[qs]{a^{ps+qr}}, \text{ Art. 224, } =a^{\frac{ps+qr}{qs}}=a^{\frac{p}{q}+\frac{r}{s}}.$$

Hence we conclude that *the exponent of any letter in a product is equal to the algebraic sum of the exponents of this letter in the* *multiplicand and multiplier, whether the exponents are positive or negative, integral or fractional.*

EXAMPLES.

1. Multiply $5a^{\frac{1}{2}}$ by $3a^{\frac{1}{3}}$. *Ans.* $15a^{\frac{5}{6}}$.
2. Multiply $21a^{\frac{1}{4}}$ by $3a^{\frac{2}{3}}$. *Ans.* $63a^{\frac{11}{12}}$.
3. Multiply $3x^{\frac{1}{2}}y^{\frac{1}{3}}$ by $4x^{\frac{1}{2}}y^{\frac{2}{3}}$.
4. Find the product of $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, and $a^{-\frac{1}{12}}$.
5. Find the product of $a^{\frac{1}{3}}$, $a^{-\frac{3}{4}}$, $a^{\frac{4}{3}}$, and $a^{\frac{1}{12}}$.

Multiplication of Polynomial Radicals.

226. By combining the preceding rules with that for the multiplication of polynomials, *Art.* 61, we may multiply together radical expressions consisting of any number of terms.

Ex. 1. Let it be required to multiply

$$a^{\frac{3}{4}}+2a^{\frac{1}{2}}-a^{\frac{1}{4}} \text{ by } a^{\frac{1}{2}}-3a^{\frac{1}{4}}+2.$$

$$\begin{array}{llllll}
a^{\frac{3}{4}}+2a^{\frac{1}{2}}-a^{\frac{1}{4}} & & & & & \\
a^{\frac{1}{2}}-3a^{\frac{1}{4}}+2 & & & & & \\
\hline
a^{\frac{5}{4}} & +2a & -a^{\frac{3}{4}} & & & \\
 & -3a & -6a^{\frac{3}{4}} & +3a^{\frac{1}{2}} & & \\
 & & +2a^{\frac{3}{4}} & +4a^{\frac{1}{2}} & -2a^{\frac{1}{4}} & \\
\hline
a^{\frac{5}{4}} & -\ a & -5a^{\frac{3}{4}} & +7a^{\frac{1}{2}} & -2a^{\frac{1}{4}}, & \textit{Ans.}
\end{array}$$

Ex. 2. Multiply $3+\sqrt{5}$ by $2-\sqrt{5}$. *Ans.* $1-\sqrt{5}$.

Ex. 3. Multiply $7+2\sqrt{6}$ by $9-5\sqrt{6}$. *Ans.* $3-17\sqrt{6}$.

Ex. 4. Multiply $9+2\sqrt{10}$ by $9-2\sqrt{10}$. *Ans.* 41.

Ex. 5. Multiply $3\sqrt{45}-7\sqrt{5}$ by $\sqrt{1\frac{4}{5}}+2\sqrt{9\frac{4}{5}}$. *Ans.* 34.

Ex. 6. Multiply $c\sqrt{a}+d\sqrt{b}$ by $c\sqrt{a}-d\sqrt{b}$.
Ans. ac^2-bd^2.

Ex. 7. Multiply $a^{\frac{7}{2}}-a^3+a^{\frac{5}{2}}-a^2+a^{\frac{3}{2}}-a+a^{\frac{1}{2}}-1$ by $a^{\frac{1}{2}}+1$.
Ans. a^4-1.

L

To divide one Radical Quantity by another.

227. The division of radical quantities depends upon the following principle:

The quotient of the nth *roots of two quantities is equal to the* nth *root of their quotient;* or,

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}},$$

for the nth power of each of these expressions is $\frac{a}{b}$, Art. 186.

Let it be required to divide $4a^2\sqrt{6by}$ by $2a\sqrt{3b}$.

$$\frac{4a^2\sqrt{6by}}{2a\sqrt{3b}}=\frac{4a^2}{2a}\sqrt{\frac{6by}{3b}}=2a\sqrt{2y},\ Ans.$$

Hence we have the following

RULE.

If necessary, reduce the given radicals to a common index. Divide the coefficient of the dividend by that of the divisor for a new coefficient; also the quantity under the radical sign in the dividend by that in the divisor, and place this quotient under the common radical sign. Then reduce the result to its simplest form.

EXAMPLES.

1. Divide $8\sqrt{108}$ by $2\sqrt{6}$. *Ans.* $12\sqrt{2}$.
2. Divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$. *Ans.* $8\sqrt[3]{4}$.
3. Divide $6\sqrt[3]{54}$ by $3\sqrt[3]{2}$. *Ans.* 6.
4. Divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.
5. Divide $4\sqrt{6a^2y}$ by $2\sqrt{3y}$.
6. Divide $16(a^3b)^{\frac{1}{m}}$ by $8(ac)^{\frac{1}{m}}$.
7. Divide $4\sqrt[3]{12}$ by $2\sqrt{3}$. *Ans.* $2\sqrt[6]{\frac{16}{3}}$.
8. Divide $\sqrt[5]{64}$ by 2. *Ans.* $\sqrt[5]{2}$.
9. Divide $\sqrt[3]{a^2bc}$ by $\sqrt[5]{ab^2c^3}$. *Ans.* $\sqrt[15]{\frac{a^7}{bc^4}}$.

228. We have seen, *Art.* 72, that *the exponent of any letter in* *a quotient is equal to the difference between the exponents of this letter in the divisor and dividend.*

The same relation holds true *whether the exponents are positive or negative, integral or fractional;* that is, universally,

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

For the quotient must be a quantity which, multiplied by the divisor, shall produce the dividend; and, according to *Art.* 225, the exponent of any letter in a product is in all cases equal to the algebraic sum of the exponents of this letter in the multiplicand and multiplier. Hence this relation must hold true universally in division.

EXAMPLES.

1. Divide $(ab)^3$ by $(ab)^{\frac{1}{2}}$. *Ans.* $(ab)^{\frac{5}{2}}$.
2. Divide $a^{\frac{2}{3}}$ by $a^{\frac{1}{4}}$.
3. Divide $4\sqrt{ab}$ by $2\sqrt[3]{ab}$. *Ans.* $2\sqrt[6]{ab}$.
4. Divide $9m^2(a-b)^{\frac{1}{3}}$ by $3m(a-b)^{\frac{1}{4}}$. *Ans.* $3m(a-b)^{\frac{1}{12}}$.
5. Divide $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{5}}$. *Ans.* $a^{\frac{1}{6}}b^{\frac{2}{15}}$.
6. Divide $4^{\frac{1}{2}}$ by $2^{\frac{3}{2}}$. *Ans.* $\frac{1}{\sqrt{2}}$.

Division of Polynomial Radicals.

229. By combining the preceding rules with that for the division of polynomials, *Art.* 80, we may divide one radical expression by another containing any number of terms.

Ex. 1. Let it be required to divide $a^{\frac{5}{6}}-a^{\frac{2}{3}}-a^{\frac{1}{2}}+a^{\frac{1}{3}}$ by $a^{\frac{1}{3}}-1$.

$$\begin{array}{r|l}
a^{\frac{5}{6}}-a^{\frac{2}{3}}-a^{\frac{1}{2}}+a^{\frac{1}{3}} & a^{\frac{1}{3}}-1 \\
\cline{2-2}
a^{\frac{5}{6}}-a^{\frac{1}{2}} \qquad\qquad & a^{\frac{1}{2}}-a^{\frac{1}{3}},\ \textit{Ans.} \\
\cline{1-1}
-a^{\frac{2}{3}}+a^{\frac{1}{3}} & \\
-a^{\frac{2}{3}}+a^{\frac{1}{3}} & \\
\cline{1-1}
\end{array}$$

Ex. 2. Divide $8a-b$ by $2a^{\frac{1}{3}}-b^{\frac{1}{3}}$.

Ans. $4a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

Ex. 3. Divide $a-41a^{\frac{1}{5}}-120$ by $a^{\frac{2}{5}}+4a^{\frac{1}{5}}+5$.

Ans. $a^{\frac{3}{5}}-4a^{\frac{2}{5}}+11a^{\frac{1}{5}}-24$.

Ex. 4. Divide $a^{\frac{1}{2}}+64b^{\frac{1}{2}}$ by $a^{\frac{1}{6}}+4b^{\frac{1}{6}}$.

Ans. $a^{\frac{1}{3}}-4a^{\frac{1}{6}}b^{\frac{1}{6}}+16b^{\frac{1}{3}}$.

Ex. 5. Divide $x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$.

Ans. $x+y$.

To involve a Radical Quantity to any power.

230. Let it be required to raise $a^{\frac{1}{m}}$ to the nth power.

The square of $a^{\frac{1}{m}}$ is $a^{\frac{1}{m}}\times a^{\frac{1}{m}}=a^{\frac{1}{m}+\frac{1}{m}}=a^{\frac{2}{m}}$.

The cube of $a^{\frac{1}{m}}$ is $a^{\frac{1}{m}}\times a^{\frac{1}{m}}\times a^{\frac{1}{m}}=a^{\frac{1}{m}+\frac{1}{m}+\frac{1}{m}}=a^{\frac{3}{m}}$.

The nth power of $a^{\frac{1}{m}}$ is $a^{\frac{1}{m}}\times a^{\frac{1}{m}}\times a^{\frac{1}{m}}$, etc., $=a^{\frac{1}{m}+\frac{1}{m}+\frac{1}{m}, \text{ etc.,}}=a^{\frac{n}{m}}$.
Hence, to involve a radical quantity to any power, we have the following

RULE.

Multiply the fractional exponent of the quantity by the exponent of the required power. If the radical has a coefficient, let this be involved separately; then reduce the result to its simplest form.

If the quantity is under the radical sign, it is generally most convenient to substitute for this sign the equivalent fractional exponent; but if we choose to retain the radical sign, we must raise the quantity under it to the required power.

EXAMPLES.

1. Required the fourth power of $\frac{2}{3}a^{\frac{1}{3}}$. *Ans.* $\frac{16}{81}a^{\frac{4}{3}}$.
2. Required the cube of $\frac{2}{3}\sqrt{3}$. *Ans.* $\frac{8}{9}\sqrt{3}$.
3. Required the square of $3\sqrt[3]{3}$.
4. Required the cube of $17\sqrt{21}$.
5. Required the fourth power of $\frac{1}{6}\sqrt{6}$. *Ans.* $\frac{1}{36}$.
6. Required the fourth power of $2\sqrt[6]{3a^4b}$.

   Ans. $16a^2\sqrt[3]{9a^2b^2}$.
7. Required the fourth power of $ab\sqrt{ab}$. *Ans.* a^6b^6.

8. Required the sixth power of $(a+b)^{\frac{1}{3}}$.

Ans. $a^2+2ab+b^2$.

9. Required the value of $\sqrt{(\frac{16}{25})^7}\times\sqrt{(\frac{25}{64})^6}$. *Ans.* $\frac{1}{80}$.

10. Required the value of $\sqrt[3]{(4ab^2)^x}\times\sqrt[3]{(2a^2b)^x}$.

Ans. $(2ab)^x$.

To Extract any Root of a Radical Quantity.

231. A root of a quantity is a factor which, multiplied by itself a certain number of times, will produce the given quantity. But we have seen that the nth power of $a^{\frac{1}{m}}$ is $a^{\frac{n}{m}}$. Therefore the nth root of $a^{\frac{n}{m}}$ is $a^{\frac{1}{m}}$. Hence we derive the following

RULE.

Divide the fractional exponent of the quantity by the index of the required root. If the radical has a coefficient, extract its root separately if possible; otherwise introduce it under the radical sign. Then reduce the result to its simplest form.

If the quantity is under the radical sign, and we choose to retain the sign, we must, if possible, extract the required root of the quantityunder the radical sign; otherwise we must multiply the index of the radical by the index of the required root.

EXAMPLES.

1. Find the square root of $9(3)^{\frac{1}{3}}$. *Ans.* $3\sqrt[6]{3}$.
2. Find the cube root of $\frac{1}{8}\sqrt{2}$. *Ans.* $\frac{1}{2}\sqrt[6]{2}$.
3. Find the square root of 10^3.
4. Find the cube root of $\frac{8}{27}a^4$.
5. Find the fourth root of $\frac{16}{81}a^{\frac{2}{3}}$.
6. Find the cube root of $\frac{64}{125}a^6$.
7. Find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$. *Ans.* $\sqrt{\frac{a}{3}}$.
8. Find the square root of $3\sqrt[3]{5}$. *Ans.* $\sqrt[6]{135}$.
9. Find the fourth root of $\frac{4}{9}\sqrt[3]{\frac{4}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{12}$.

Operations on Imaginary Quantities.

232. It has been shown, Art. 195, that an even root of a negative quantity is impossible. Thus, $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-5a}$ are algebraic symbols representing operations which it is impossible to execute; for the square of every quantity, whether positive or negative, is necessarily positive. Quantities of this nature are called *imaginary* or *impossible* quantities. Nevertheless, such expressions do frequently occur, and it is necessary to establish proper rules for operating upon them.

233. *The square root of a negative quantity may always be represented by the square root of a positive quantity multiplied by the square root of* -1.

Thus,
$$\sqrt{-4}=\sqrt{4\times -1}=2\sqrt{-1},$$
$$\sqrt{-3}=\sqrt{3\times -1}=\sqrt{3}\sqrt{-1},$$
$$\sqrt{-a}=\sqrt{a\times -1}=\sqrt{a}\sqrt{-1}.$$

The factor $\sqrt{-1}$ is called the *imaginary factor*, and the other factor is called its *coefficient.*

234. When several imaginary factors are to be multiplied together, it is best to resolve each of them into two factors, of which one is the square root of a positive quantity, and the other $\sqrt{-1}$. We can then multiply together the coefficients of the imaginary factor by methods already explained. It only remains to deduce a rule for multiplying the imaginary factor into itself; that is, for raising the imaginary factor to a power whose exponent is equal to the number of factors.

The first power of $\sqrt{-1}$ is $\sqrt{-1}$.

The second power, by the definition of square root, is -1.

The third power is the product of the first and second powers, or $-1\times\sqrt{-1}=-\sqrt{-1}$.

The fourth power is the square of the second, or $+1$.

The fifth is the product of the first and fourth; that is, it is the same as the first; the sixth is the same as the second, and so on; so that all the powers of $\sqrt{-1}$ form a repeating cycle of the following terms:

$$+\sqrt{-1},\ -1,\ -\sqrt{-1},\ +1.$$

EXAMPLES.

1. Multiply $\sqrt{-9}$ by $\sqrt{-4}$.

$\sqrt{-9}\times\sqrt{-4}=3\sqrt{-1}\times2\sqrt{-1}=6\sqrt{(-1)^2}=-6$, *Ans.*

2. Multiply $1+\sqrt{-1}$ by $1-\sqrt{-1}$. *Ans.* 2.
3. Multiply $\sqrt{18}$ by $\sqrt{-2}$.
4. Multiply $5+2\sqrt{-3}$ by $2-\sqrt{-3}$.
5. Multiply $a\sqrt{-b}$ by $c\sqrt{-d}$. *Ans.* $-ac\sqrt{bd}$.
6. Multiply $1-\sqrt{-1}$ by itself. *Ans.* $-2\sqrt{-1}$.
7. Multiply $2\sqrt{3}-\sqrt{-5}$ by $4\sqrt{3}-2\sqrt{-5}$. *Ans.* $14-8\sqrt{-15}$.
8. Multiply $a+\sqrt{b}\sqrt{-1}$ by $a-\sqrt{b}\sqrt{-1}$. *Ans.* a^2+b.
9. Multiply $a\sqrt{-a^2b^3}$ by $\sqrt{-a^4b^5}$.
10. Multiply $\sqrt{-a}+\sqrt{-b}$ by $\sqrt{-a}-\sqrt{-b}$.
11. Multiply $\sqrt{-17}+\sqrt{-19}$ by $\sqrt{-119}-\sqrt{-133}$. *Ans.* $2\sqrt{7}$.

235. *Division of Imaginary Quantities.*—The quotient of one imaginary term divided by another is easily found by resolving both terms into factors, as in the preceding article.

Ex. 1. Let it be required to divide $\sqrt{-ab}$ by $\sqrt{-a}$.

$$\frac{\sqrt{-ab}}{\sqrt{-a}}=\frac{\sqrt{ab}\sqrt{-1}}{\sqrt{a}\sqrt{-1}}=\sqrt{b},\ \textit{Ans.}$$

Ex. 2. Divide $b\sqrt{-1}$ by $c\sqrt{-1}$. *Ans.* $\frac{b}{c}$.

Ex. 3. Divide unity by $\sqrt{-1}$. *Ans.* $-\sqrt{-1}$.

Ex. 4. Divide a by $b\sqrt{-1}$.

Ex. 5. Divide a by $\sqrt{a}\sqrt{-1}$.

Ex. 6. Divide $\sqrt{-12}+\sqrt{-6}+\sqrt{-9}$ by $\sqrt{-3}$.

Ex. 7. Divide $2\sqrt{8}-\sqrt{-10}$ by $-\sqrt{-2}$.

Ans. $\sqrt{5}+4\sqrt{-1}$.

To find Multipliers which shall cause Surds to become Rational.

236. 1st. When the surd is a *monomial.*

The quantity $\sqrt{a}$ is rendered rational by multiplying it by $\sqrt{a}$.

For $\sqrt{a}\times\sqrt{a}=a^{\frac{1}{2}}\times a^{\frac{1}{2}}=a$.

So, also, $a^{\frac{1}{3}}$ is rendered rational by multiplying it by $a^{\frac{2}{3}}$. Also, $a^{\frac{1}{4}}$ is rendered rational by multiplying it by $a^{\frac{3}{4}}$; and $a^{\frac{1}{n}}$ by multiplying it by $a^{1-\frac{1}{n}}$.

Hence we deduce the following

RULE.

Multiply the surd by the same quantity having such an exponent as, when added to the exponent of the given surd, shall make unity.

237. 2d. When the surd is a *binomial.*

If the binomial contains only the *square root, multiply the given binomial by the same terms connected by the opposite sign, and it will give a rational product.*

Thus the expression $\sqrt{a}+\sqrt{b}$ multiplied by $\sqrt{a}-\sqrt{b}$ gives for a product $a-b$.

Also the expression $\sqrt[4]{a}+\sqrt[4]{b}$ multiplied by $\sqrt[4]{a}-\sqrt[4]{b}$ gives for a product $\sqrt{a}-\sqrt{b}$, which may be rendered rational by multiplying it by $\sqrt{a}+\sqrt{b}$.

In general, $\sqrt[m]{a}\pm\sqrt[n]{b}$ may be rendered rational by successive multiplications whenever m and n denote any power of 2. When m and n are not powers of 2, the binomial may still be rendered rational by multiplication, but the process becomes more complicated.

Ex. 1. Find a multiplier which shall render $\sqrt{5}+\sqrt{3}$ rational, and determine the product.

Ex. 2. Find a multiplier which shall render $\sqrt{3}-\sqrt{x}$ rational, and determine the product.

Ex. 3. Find multipliers which shall render $\sqrt{3}-\sqrt[4]{x}$ rational, and determine the product.

238. 3d. When the surd is a *trinomial.*

When a trinomial surd contains only radicals of the second degree, we may reduce it to a binomial surd by multiplying it by the same expression, with the sign of one of the terms changed. Thus, $\sqrt{a}+\sqrt{b}+\sqrt{c}$ multiplied by $\sqrt{a}+\sqrt{b}-\sqrt{c}$ gives for a product $a+b-c+2\sqrt{ab}$, which may be put under the form of $m+2\sqrt{ab}$.

Ex. 1. Find multipliers that shall make $\sqrt{5}+\sqrt{3}-\sqrt{2}$ rational, and determine the product.

Ex. 2. Find multipliers that shall make $1+\sqrt{2}+\sqrt{3}$ rational, and determine the product.

To transform a Fraction whose Denominator is a Surd in such a Manner that the Denominator shall be Rational.

239. If we have a radical expression of the form $\dfrac{a}{\sqrt{b}+\sqrt{c}}$ or $\dfrac{a}{\sqrt{b}-\sqrt{c}}$, it may be transformed into an equivalent expression in which the denominator is rational by multiplying both terms of the fraction by $\sqrt{b}\pm\sqrt{c}$. Hence the

RULE.

Multiply both numerator and denominator by a factor which will render the denominator rational.

EXAMPLES.

Reduce the following fractions to equivalent fractions having a rational denominator:

1. $\dfrac{2}{\sqrt{3}}$. *Ans.* $\dfrac{2\sqrt{3}}{3}$.

2. $\dfrac{1}{\sqrt{5}-\sqrt{2}}$. *Ans.* $\dfrac{\sqrt{5}+\sqrt{2}}{3}$.

3. $\dfrac{\sqrt{2}}{3-\sqrt{2}}$. *Ans.* $\dfrac{3\sqrt{2}+2}{7}$.

4. $\dfrac{a-\sqrt{b}}{a+\sqrt{b}}$. $\dfrac{a^2-2a\sqrt{b}+b}{a^2-b}$.

5. $\dfrac{4}{\sqrt{3}+\sqrt{2}+1}$. *Ans.* $2+\sqrt{2}-\sqrt{6}$.

6. $\dfrac{a}{\sqrt{b}}$.

7. $\dfrac{a}{\sqrt[3]{b}}$.

8. $\dfrac{\sqrt{2}}{\sqrt[3]{2}}$. *Ans.* $\sqrt[6]{2}$.

9. $\dfrac{1+a+\sqrt{1-a^2}}{1+a-\sqrt{1-a^2}}$. *Ans.* $\dfrac{1+\sqrt{1-a^2}}{a}$.

240. The *utility* of the preceding transformations will be seen if we attempt to compute the *numerical value* of a fractional surd.

Ex. 1. Let it be required to find the square root of $\frac{3}{7}$; that is, to find the value of the fraction $\dfrac{\sqrt{3}}{\sqrt{7}}$.

Making the denominator rational, we have $\dfrac{\sqrt{21}}{7}$, and the value of the fraction is found to be 0.6546.

Ex. 2. Compute the value of the fraction $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$.

Ans. 3.1003.

Ex. 3. Compute the value of the fraction $\dfrac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$.

Ans. 0.5595.

Ex. 4. Compute the value of the fraction $\dfrac{\sqrt{3}}{2\sqrt{8}+3\sqrt{5}-7\sqrt{2}}$.

Ans. 0.7025.

Ex. 5. Compute the value of the fraction $\frac{9+2\sqrt{10}}{9-2\sqrt{10}}$.

Ans. 5.7278.

Square Root of a Binomial Surd.

241. A *binomial surd* is a binomial, one or both of whose terms are surds, as $2+\sqrt{3}$ and $\sqrt{5}-\sqrt{2}$.

A *quadratic surd* is the square root of an imperfect square.

If we square the binomial surd $2+\sqrt{3}$, we shall obtain $7+4\sqrt{3}$. Hence the square root of $7+4\sqrt{3}$ is $2+\sqrt{3}$; that is, *a binomial surd of the form* $a\pm\sqrt{b}$ *may sometimes be a perfect square.*

242. The method of extracting the square root of an expression of the form $a\pm\sqrt{b}$ is founded upon the following principles:

1st. *The sum or difference of two quadratic surds can not be equal to a rational quantity.*

Let $\sqrt{a}$ and $\sqrt{b}$ denote two surd quantities, and, if possible, suppose

$$\sqrt{a}\pm\sqrt{b}=c,$$

where c denotes a rational quantity. By transposing $\sqrt{b}$ and squaring both members, we obtain

$$\pm\sqrt{b}=\frac{b+c^2-a}{2c}.$$

The second member of the equation contains only rational quantities, while $\sqrt{b}$ was supposed to be irrational; that is, we have an irrational quantity equal to a rational one, which is impossible. Hence the sum or difference of two quadratic surds can not be equal to a rational quantity.

243. 2d. *In any equation which involves both rational quantities and quadratic surds, the rational parts in the two members are equal, and also the irrational parts.*

Suppose we have

$$x+\sqrt{y}=a+\sqrt{b}.$$

Then, if x be not equal to a, suppose it to be equal to $a+m$; then

$$a+m+\sqrt{y}=a+\sqrt{b},$$

so that
$$m=\sqrt{b}-\sqrt{y};$$

that is, a rational quantity is equal to the difference of two quadratic surds, which, by the last article, is impossible. Therefore $x=a$, and consequently $\sqrt{y}=\sqrt{b}$.

244. *To find an expression for the square root of* $a\pm\sqrt{b}$.

Let us assume $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$. (1.)

By squaring, $a+\sqrt{b}=x+2\sqrt{xy}+y$. (2.)

By Art. 243, $a=x+y$, (3.)

and $\sqrt{b}=2\sqrt{xy}$. (4.)

Subtracting (4) from (3), we have

$$a-\sqrt{b}=x-2\sqrt{xy}+y. \quad (5.)$$

By evolution, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$. (6.)

Multiplying (1) by (6), we have

$$\sqrt{a^2-b}=x-y. \quad (7.)$$

Adding (3) and (7), $a+\sqrt{a^2-b}=2x$. (8.)

Hence, $x=\dfrac{a+\sqrt{a^2-b}}{2}$. (9.)

Subtracting (7) from (3),

$$y=\frac{a-\sqrt{a^2-b}}{2}. \quad (10.)$$

It is obvious from these equations that x and y will be rational *when* a^2-b *is a perfect square.* If a^2-b be not a perfect square, the values of $\sqrt{x}$ and $\sqrt{y}$ will be complex surds.

Hence, to obtain the square root of a binomial surd, we proceed as follows:

Let a represent the rational part, and $\sqrt{b}$ the radical part, and find the values of x and y in equations (9) and (10). Then, if the binomial is of the form $a+\sqrt{b}$, its square root will be $\sqrt{x}+\sqrt{y}$. If the binomial is of the form $a-\sqrt{b}$, its square root will be $\sqrt{x}-\sqrt{y}$.

EXAMPLES.

1. Required the square root of $4+2\sqrt{3}$.

Here $a=4$ and $\sqrt{b}=2\sqrt{3}$; or $b=12$.

Hence $$x=\frac{4+\sqrt{16-12}}{2}=3,$$

$$y=\frac{4-\sqrt{16-12}}{2}=1.$$

Hence $\sqrt{x}+\sqrt{y}=\sqrt{3}+1$, *Ans.*

Verification. The square of $\sqrt{3}+1$ is $3+2\sqrt{3}+1=4+2\sqrt{3}$.

2. Required the square root of $11+6\sqrt{2}$.

Here $a=11$ and $\sqrt{b}=6\sqrt{2}$; or $b=72$.

$$x=9 \text{ and } y=2.$$

$$\sqrt{x}+\sqrt{y}=3+\sqrt{2}, \textit{ Ans.}$$

3. Required the square root of $11-2\sqrt{30}$. *Ans.* $\sqrt{6}-\sqrt{5}$.

4. Required the square root of $2+\sqrt{3}$. *Ans.* $\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}$.

5. Required the square root of $7+2\sqrt{10}$. *Ans.* $\sqrt{5}+\sqrt{2}$.

6. Required the square root of $18+8\sqrt{5}$.

Ans. $\sqrt{10}+2\sqrt{2}$.

245. This method is applicable even when the binomial contains imaginary quantities.

7. Required the square root of $1+4\sqrt{-3}$.

Here $a=1$ and $\sqrt{b}=4\sqrt{-3}$.

Hence $b=-48$ and $a^2-b=49$.

Therefore $x=4$ and $y=-3$.

The required square root is therefore $2+\sqrt{-3}$, *Ans.*

8. Required the square root of $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$.

Ans. $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$.

9. Required the square root of $2\sqrt{-1}$ or $0+2\sqrt{-1}$.

Ans. $1+\sqrt{-1}$.

10. Required the value of the expression

$$\sqrt{6+2\sqrt{5}}-\sqrt{6-2\sqrt{5}}.$$

11. Required the value of the expression

$$\sqrt{4+3\sqrt{-20}}+\sqrt{4-3\sqrt{-20}}. \qquad \textit{Ans. } 6.$$

12. Required the square root of $-3+\sqrt{-16}$.

Ans. $1+2\sqrt{-1}$.

13. Required the square root of $8\sqrt{-1}$.

Ans. $2+2\sqrt{-1}$.

Simple Equations containing Radical Quantities.

246. When the unknown quantity is affected by the radical sign, we must first render the terms containing the unknown quantity rational. This may generally be done by successive involutions. For this purpose we first free the equation from fractions. If there is but *one* radical expression, we bring that to stand alone on one side of the equation, and involve both members to a power denoted by the index of the radical.

Ex. 1. Given $x+\sqrt{x^2-3x+60}=12$ to find x.

Transposing x and squaring each member, we have

$$x^2-3x+60=144-24x+x^2;$$

whence $$x=4.$$

Ex. 2. Given $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$ to find x.

Ans. $x=\dfrac{a}{3}$.

Ex. 3. Given $\dfrac{17-5\sqrt{x}}{11}=-3$ to find x.

Ex. 4. Given $\sqrt{4x^2-7x-6}=9-2x$ to find x.

247. If the equation contains *two* radical expressions combined with other terms which are rational, it is generally best to bring one of the radicals to stand alone on one side of the equation before involution. One of the radicals will thus be made to disappear, and, by repeating the operation, the remaining radical may be exterminated.

Ex. 5. Given $\sqrt{x+19}+\sqrt{x+10}=9$ to find x.

By transposition, $\sqrt{x+19}=9-\sqrt{x+10}$.

Squaring, $x+19=91+x-18\sqrt{x+10}$.

Transposing and reducing,

$$\sqrt{x+10}=4.$$

Squaring, $x+10=16$;

whence $x=6$.

Ex. 6. Given $\sqrt{36+x}=18+\sqrt{x}$ to find x.

Ex. 7. Given $\sqrt{x+4ab}=2b+\sqrt{x}$ to find x.

Ex. 8. Given $x=\sqrt{a^2+x\sqrt{b^2+x^2-a^2}}+a$ to find x.

248. When an equation contains a fraction involving radical quantities in both numerator and denominator, it is sometimes best to render the denominator rational by *Art.* 239; but the best method can only be determined by trial.

Ex. 9. Given $\dfrac{\sqrt{x}+\sqrt{x-3}}{\sqrt{x}-\sqrt{x-3}}=\dfrac{3}{x-3}$ to find x.

Multiply both terms of the first fraction by $\sqrt{x}+\sqrt{x-3}$, and we have

$$\frac{(\sqrt{x}+\sqrt{x-3})^2}{x-(x-3)}=\frac{3}{x-3},$$

or

$$(\sqrt{x}+\sqrt{x-3})^2=\frac{9}{x-3}.$$

Extracting the square root,

$$\sqrt{x}+\sqrt{x-3}=\frac{3}{\sqrt{x-3}}.$$

Clearing of fractions,

$$\sqrt{x^2-3x}+x-3=3;$$

whence $x=4$, *Ans.*

Ex. 10. Given $\dfrac{\sqrt{9x+13}+\sqrt{9x}}{\sqrt{9x+13}-\sqrt{9x}}=13$ to find x.

Ex. 11. Given $\sqrt[m]{ax+b}=\sqrt[m]{cx+d}$ to find x.

Ex. 12. Given $\dfrac{50\sqrt[10]{x+24}-9}{3+5\sqrt[10]{x+24}}=7$ to find x.

Ex. 13. Given $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{1}{2}(\sqrt{3x}-1)$ to find x.

Ans. $x=3$.

Ex. 14. Given $\frac{\sqrt{x}+4m}{\sqrt{x}+3n}=\frac{\sqrt{x}+2m}{\sqrt{x}+n}$ to find x.

Ans. $x=\left(\frac{mn}{m-n}\right)^2$.

Ex. 15. Given $(\sqrt{9x}-6)(\sqrt{x}+25)=(5+3\sqrt{x})(\sqrt{x}+3)$ to find x.

Ans. $x=9$.

Ex. 16. Given $\sqrt{2x-3n}=3\sqrt{n}-\sqrt{2x}$ to find x.

Ans. $x=2n$.

Ex. 17. Given $\frac{\sqrt{9x}-4}{\sqrt{x}+2}=\frac{15+\sqrt{9x}}{\sqrt{x}+40}$ to find x.

Ex. 18. Given $\frac{\sqrt{6x}-2}{\sqrt{6x}+2}=\frac{\sqrt{6x}-9}{\sqrt{6x}+6}$ to find x.

Ex. 19. Given $\frac{5x-9}{\sqrt{5x}+3}-1=\frac{\sqrt{5x}-3}{2}$ to find x.

Ex. 20. Given $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$ to find x.

Ans. $x=\frac{(a-b)^2}{2a-b}$.

CHAPTER XIV.

EQUATIONS OF THE SECOND DEGREE.

249. An *equation of the second degree*, or a *quadratic equation* with one unknown quantity, is one in which the highest power of the unknown quantity is a square.

250. Every equation of the second degree containing but one unknown quantity can be reduced to *three terms;* one containing the *second power* of the unknown quantity, another the *first power*, and the third a *known quantity;* that is, it can be reduced to the form

$$x^2+px=q.$$

Suppose we have the equation

$$\frac{3x^2}{4}+\frac{7x-6}{12}+\frac{x-2}{3}+13\tfrac{2}{3}=(x+3)(x-1).$$

Clearing of fractions and expanding, we have

$$9x^2+7x-6+4x-8+164=12x^2+24x-36.$$

Transposing and uniting similar terms, we have

$$-3x^2-13x=-186.$$

Dividing by the coefficient of x^2, that is, by -3, we have

$$x^2+\frac{13}{3}x=62,$$

which is of the form above given, p in this case being equal to $\frac{13}{3}$, and q being equal to 62.

251. In order *to reduce a quadratic equation to three terms*, we must first clear it of fractions, and perform all the operations indicated. We then transpose all the terms which contain the unknown quantity to the first member of the equation, and the known quantities to the second member; unite similar terms, and divide each term of the resulting equation by the coefficient of x^2.

252. An equation of the form $x^2+px=q$ is called a *complete equation* of the second degree, because it contains each class of terms of which the general equation is susceptible.

253. The coefficient of the first power of the unknown quantity may reduce to zero, in which case the equation is said to be *incomplete.*

An *incomplete equation* of the second degree, when reduced, contains but two terms: one containing the square of the unknown quantity, and the other a known term.

Incomplete Equations of the Second Degree.

254. An incomplete equation of the second degree may be reduced to the form $$x^2=q.$$

Extracting the square root of each member, we have
$$x=\pm\sqrt{q}.$$

If q be a positive number, either integral or fractional, we can extract its square root, either exactly or approximately, by the rules of Chap. XII. Hence, to solve an incomplete equation of the second degree, we have the

RULE.

Reduce the equation to the form $x^2=q$, and extract the square root of each member of the equation.

255. Since the square of both $+m$ and $-m$ is $+m^2$, the square of $+\sqrt{q}$ and that of $-\sqrt{q}$ are both $+q$. Hence the above equation is susceptible of *two solutions*, or has *two roots;* that is, there are two quantities, which, when substituted for x in the original equation, will render the two members identical. These are $$+\sqrt{q} \text{ and } -\sqrt{q}.$$

Hence, *Every incomplete equation of the second degree has two roots, equal in numerical value, but with opposite signs.*

EXAMPLES.

Find the values of x in each of the following equations:

1. $4x^2-7=3x^2+9.$ *Ans.* $x=\pm 4.$

Show that each of these values will satisfy the equation.

2. $x^2-17=130-2x^2$. *Ans.* $x=\pm 7$.

3. $\frac{4x^2+5}{9}=45$.

4. $x^2+ab=5x^2$. *Ans.* $x=\frac{\pm\sqrt{ab}}{2}$.

5. $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$. *Ans.* $x=\pm\frac{a}{\sqrt{3}}$.

6. $ax^2-5c=bx^2-3c+d$.

7. $\frac{x^2}{3}-3+\frac{5x^2}{12}=\frac{7}{24}-x^2+\frac{299}{24}$.

8. $12ab+x^2=4a^2+9b^2$.

9. $11-\frac{x+25}{x^2}=3-\frac{x-25}{x^2}$.

10. $x+\sqrt{x^2-17}=\frac{4}{\sqrt{x^2-17}}$. *Ans.* $x=\pm 4\frac{1}{5}$.

11. $\frac{x+a}{x-a}+\frac{x-a}{x+a}=\frac{2(a^2+1)}{(1+a)(1-a)}$. *Ans.* $x=\pm 1$.

12. $\frac{x^2+3x-7}{x+2+\frac{18}{x}}=1$. *Ans.* $x=\pm 3$.

Note. Clearing of fractions and transposing, we find in each member of this equation a binomial factor, which being canceled, the equation is easily solved.

PROBLEMS.

Prob. 1. What two numbers are those whose sum is to the greater as 10 to 7, and whose sum, multiplied by the less, produces 270?

Let $10x=$ their sum.
Then $7x=$ the greater number,
and $3x=$ the less.
Whence $30x^2=270$,
and $x^2=9$;
therefore $x=\pm 3$,
and the numbers are ± 21 and ± 9.

Prob. 2. What two numbers are those whose sum is to the greater as m to n, and whose sum, multiplied by the less, is equal to a?

Ans. $\pm\sqrt{\dfrac{an^2}{m(m-n)}}$ and $\pm\sqrt{\dfrac{a(m-n)}{m}}$.

Prob. 3. What number is that, the third part of whose square being subtracted from 20, leaves a remainder equal to 8?

Prob. 4. What number is that, the mth part of whose square being subtracted from a, leaves a remainder equal to b?

Ans. $\pm\sqrt{m(a-b)}$.

Prob. 5. Find three numbers in the ratio of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$, the sum of whose squares is 724.

Prob. 6. Find three numbers in the ratio of m, n, and p, the sum of whose squares is equal to a.

Ans.

$$\pm\sqrt{\frac{am^2}{m^2+n^2+p^2}};\ \pm\sqrt{\frac{an^2}{m^2+n^2+p^2}};\ \text{and}\ \pm\sqrt{\frac{ap^2}{m^2+n^2+p^2}}.$$

Prob. 7. Divide the number 49 into two such parts that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Ans. 21 and 28.

Note. In solving this Problem, it is necessary to assume a principle employed in Arithmetic, viz., If four quantities are proportional, *the product of the extremes is equal to the product of the means.*

Thus, if $a : b :: c : d$,

then $ad=bc$.

Prob. 8. Divide the number a into two such parts that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as m to n.

Ans. $\dfrac{a\sqrt{m}}{\sqrt{m}+\sqrt{n}}$ and $\dfrac{a\sqrt{n}}{\sqrt{m}+\sqrt{n}}$.

Prob. 9. There are two square grass-plats, a side of one of which is 10 yards longer than a side of the other, and their areas are as 25 to 9. What are the lengths of the sides?

Prob. 10. There are two squares whose areas are as m to n, and a side of one exceeds a side of the other by a. What are the lengths of the sides?

Ans. $\frac{a\sqrt{m}}{\sqrt{m}-\sqrt{n}}$ and $\frac{a\sqrt{n}}{\sqrt{m}-\sqrt{n}}$.

Prob. 11. Two travelers, A and B, set out to meet each other, A leaving Hartford at the same time that B left New York. On meeting, it appeared that A had traveled 18 miles more than B, and that A could have gone B's journey in $15\frac{3}{4}$ hours, but B would have been 28 hours in performing A's journey. What was the distance between Hartford and New York?

Ans. 126 miles.

Prob. 12. From two places at an unknown distance, two bodies, A and B, move toward each other, A going a miles more than B. A would have described B's distance in n hours, and B would have described A's distance in m hours. What was the distance of the two places from each other?

Ans. $a\times\frac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}$.

Prob. 13. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons, and then, filling the vessel with water, draws off the same quantity of liquor as before, and so on for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Ans. 64, 48, 36, and 27 gallons.

Note. Suppose $\frac{1}{x}$ part is drawn each time.

Then $256-\frac{256}{x}=\frac{256(x-1)}{x}$ remains after the first draught.

Similarly, $\frac{256(x-1)^2}{x^2}$ remains after the second draught, and so on.

Hence $$\frac{256(x-1)^4}{x^4}=81.$$

Prob. 14. A number a is diminished by the nth part of itself, this remainder is diminished by the nth part of itself, and so on to the fourth remainder, which is equal to b. Required the value of n.

$$Ans.\ \frac{\sqrt[4]{a}}{\sqrt[4]{a}-\sqrt[4]{b}}.$$

Prob. 15. Two workmen, A and B, were engaged to work for a certain number of days at different rates. At the end of the time, A, who had played 4 of those days, received 75 shillings, but B, who had played 7 of those days, received only 48 shillings. Now had B only played 4 days and A played 7 days, they would have received the same sum. For how many days were they engaged? *Ans.* 19 days.

Prob. 16. A person employed two laborers, allowing them different wages. At the end of a certain number of days, the first, who had played a days, received m shillings, and the second, who had played b days, received n shillings. Now if the second had played a days, and the other b days, they would both have received the same sum. For how many days were they engaged?

$$Ans.\ \frac{b\sqrt{m}-a\sqrt{n}}{\sqrt{m}-\sqrt{n}} \text{ days.}$$

Complete Equations of the Second Degree.

256. In order to solve a complete equation of the second degree, let the equation be reduced to the form

$$x^2+px=q.$$

If we can by any transformation render the first member of this equation the perfect square of a binomial, we can reduce the equation to one of the first degree by extracting its square root.

Now we know that the square of a binomial, $x+a$, or $x^2+2ax+a^2$, is composed of the square of the first term, plus twice the product of the first term by the second, plus the square of the second term.

Hence, considering x^2+px as the first two terms of the square of a binomial, and consequently px as being twice the product of the first term of the binomial by the second, it is evident that the second term of this binomial must be $\frac{p}{2}$.

257. In order, therefore, that the expression x^2+px may be rendered a perfect square, we must add to it the square of this second term $\frac{p}{2}$; and in order that the equality of the two members may not be destroyed, we must add the same quantity to the second member of the equation. We shall then have

$$x^2+px+\frac{p^2}{4}=q+\frac{p^2}{4}.$$

Taking the square root of each member, we have

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}};$$

whence, by transposition, $x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}$.

Thus the equation has *two roots:* one corresponding to the *plus* sign of the radical, and the other to the *minus* sign. These two roots are

$$-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}} \text{ and } -\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

258. Hence, for solving a complete equation of the second degree, we have the following

RULE.

1st. *Reduce the given equation to the form of* $x^2+px=q$.

2d. *Add to each member of the equation the square of half the coefficient of the first power of x.*

3d. *Extract the square root of both members, and the equation will be reduced to one of the first degree, which may be solved in the usual manner.*

259. When the equation has been reduced to the form $x^2+px=q$, its two roots will be equal to *half the coefficient of the sec-*

ond term, taken with a contrary sign, plus or minus the square root of the second member, increased by the square of half the coefficient of the second term.

Ex. 1. Let it be required to solve the equation

$$x^2-10x=-16.$$

Completing the square by adding to each member the square of half the coefficient of the second term, we have

$$x^2-10x+25=25-16=9.$$

Extracting the root, $x-5=\pm 3.$

Whence $x=5\pm 3=\ 8$ or 2, *Ans.*

To verify these values of x, substitute them in the original equation, and we shall have

$$8^2-10\times 8=64-80=-16.$$

Also, $2^2-10\times 2=\ 4-20=-16.$

Ex. 2. Solve the equation $2x^2+8x-20=70.$

Ans. $x=+5$ or -9.

Ex. 3. Solve the equation $3x^2-3x+6=5\frac{1}{3}$.

Reducing, $x^2-x=-\frac{2}{9}$.

Completing the square, $x^2-x+\frac{1}{4}=\frac{1}{4}-\frac{2}{9}=\frac{1}{36}$.

Hence $x=\frac{1}{2}\pm\frac{1}{6}=\frac{2}{3}$ or $\frac{1}{3}$, *Ans.*

Second Method of completing the Square.

260. The preceding method of completing the square is always applicable; nevertheless, it sometimes gives rise to inconvenient fractions. In such cases the following method may be preferred. Let the equation be reduced to the form

$$ax^2+bx=c,$$

in which a and b are whole numbers, and *prime* to each other, but c may be either entire or fractional.

Multiply each member of this equation by $4a$, and it becomes

$$4a^2x^2+4abx=4ac.$$

Adding b^2 to each member, we have

$$4a^2x^2+4abx+b^2=4ac+b^2,$$

where the first member is a complete square, and its terms are entire.

Extracting the square root, we have

$$2ax+b=\pm\sqrt{4ac+b^2}.$$

Transposing b, and dividing by $2a$,

$$x=\frac{-b\pm\sqrt{4ac+b^2}}{2a},$$

which is the same result as would be obtained by the former rule; but by this method we have avoided the introduction of fractions in completing the square.

If b is an even number, $\frac{b}{2}$ will be an entire number; and it would have been sufficient to multiply each member by a, and add $\frac{b^2}{4}$ to each member. Hence we have the following

RULE.

1st. *Reduce the equation to the form* $ax^2+bx=c$, *where* a *and* b *are prime to each other.*

2d. *If* b *is an odd number, multiply the equation by four times the coefficient of* x^2, *and add to each member the square of the coefficient of* x.

3d. *If* b *is an even number, multiply the equation by the coefficient of* x^2, *and add to each member the square of half the coefficient of* x.

Ex. 4. Solve the equation $6x^2-13x=-6$.

Multiplying by 4×6, and adding 13^2 to each member, we have $144x^2-312x+169=169-144=25$.

Extracting the root, $12x-13=\pm5$.

Whence $12x=18$ or 8,

and $x=\frac{3}{2}$ or $\frac{2}{3}$.

Ex. 5. Solve the equation $110x^2-21x=-1$.

Multiplying by 440, and adding 21^2 to each member, we have

$$48400x^2-9240x+441=1.$$

Extracting the root, $220x-21=\pm1$.

Whence $x=\frac{1}{10}$ or $\frac{1}{11}$.

Ex. 6. Solve the equation $7x^2-3x=160$.

Ans. $x=5$ or $-\frac{32}{7}$.

261. *Modification of the preceding Method.*—The preceding method sometimes gives rise to numbers which are unnecessarily large. When the equation has been reduced to the form $ax^2+bx=c$, it is sufficient to multiply it by any number which will render the first term a perfect square. Let the resulting equation be

$$m^2x^2+nx=q.$$

The first member will become a complete square by the addition of $\left(\frac{n}{2m}\right)^2$, and the equation will then be

$$m^2x^2+nx+\frac{n^2}{4m^2}=q+\frac{n^2}{4m^2}.$$

This method is expressed in the following

RULE.

Having reduced the equation to the form $ax^2+bx=c$, multiply the equation by any quantity (the least possible) which will render the first term a perfect square. Divide the coefficient of x in this new equation by twice the square root of the coefficient of x^2, and add the square of this result to both members.

Ex. 7. Solve the equation $8x^2+9x=99$.

$$16x^2+18x+\left(\frac{9}{4}\right)^2=\frac{3249}{16},$$

$$4x=-\frac{9}{4}\pm\frac{57}{4},$$

$$x=3 \text{ or } -\frac{33}{8}, \textit{ Ans.}$$

Ex. 8. Solve the equation $16x^2-15x=34$.

$$16x^2-15x+\left(\frac{15}{8}\right)^2=\frac{2401}{64},$$

$$4x=8 \text{ or } -\frac{17}{4},$$

$$x=2 \text{ or } -\frac{17}{16}, \textit{ Ans.}$$

Ex. 9. Solve the equation $12x^2=21+\frac{1}{4}x$.

Solve the following equations:

Ex. 10. $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$. *Ans.* $x=7$ or $-6\frac{1}{3}$.

Ex. 11. $x^2-x-40=170$. *Ans.* $x=15$ or -14.

Ex. 12. $3x^2+2x-9=76$.

Ex. 13. $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8.$

Ex. 14. $3x-\frac{6x^2-40}{2x-1}-\frac{3x-10}{9-2x}=2.$

This equation reduces to $x^2-15\frac{1}{2}x=-46.$

Ans. $x=11\frac{1}{2}$ or 4.

Ex. 15. $\frac{15}{x}-\frac{72-6x}{2x^2}=2.$ *Ans.* $x=3$ or 6.

Ex. 16. $\frac{90}{x}-\frac{90}{x+1}-\frac{27}{x+2}=0.$

Ex. 17. $x^2-x\sqrt{3}=x-\frac{1}{2}\sqrt{3}.$

Ans. $x=\frac{\sqrt{3}+3}{2}$ or $\frac{\sqrt{3}-1}{2}.$

Ex. 18. $\frac{x-1}{x-2}-\frac{x-3}{x-4}=-\frac{2}{3}.$ *Ans.* $x=5$ or 1.

Ex. 19. $\frac{x}{a+x}+\frac{a+x}{x}=\frac{5}{2}.$ *Ans.* $x=a$ or $-2a$.

Ex. 20. $x^2-(a+b)x+ab=0.$ *Ans.* $x=a$ or b.

Ex. 21. $(3x-25)(7x+29)=0.$ *Ans.* $x=8\frac{1}{3}$ or $-4\frac{1}{7}$.

Ex. 22. $\frac{3x-2}{2x-5}+\frac{2x-5}{3x-2}=\frac{10}{3}$ *Ans.* $x=\frac{13}{3}$ or $\frac{1}{7}.$

Ex. 23. $(x-1)(x-2)+(x-2)(x-4)=6(2x-5).$

Ans. $x=8$ or $\frac{5}{2}.$

Ex. 24. $\frac{170}{x}-\frac{170}{x+1}=\frac{51}{x+2}.$ *Ans.* $x=4$ or $-1\frac{2}{3}$.

Ex. 25. $\frac{a^2+ax+x^2}{a+x}+\frac{a^2-ax+x^2}{a-x}=\frac{ab}{3a-4b+x}.$

Ans. $x=-3a$ or $3a-\frac{2a^2}{b}.$

Equations which may be solved like Quadratics.

262. There are many equations of a higher degree than the second, which may be solved by methods similar to those employed for quadratics. To this class belong *all equations which contain only two powers of the unknown quantity, and in which* *the greater exponent is double the less.* Such equations are of the form

$$x^{2n}+px^n=q,$$

where n may be either integral or fractional.

For if we assume $y=x^n$, then $y^2=x^{2n}$, and this equation becomes

$$y^2+py=q;$$

whence $$x^n=y=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}.$$

Extracting the nth root of each member, we have

$$x=\sqrt[n]{-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}}}.$$

Ex. 1. Solve the equation $x^4-13x^2=-36$.

Assuming $x^2=y$, the above becomes

$$y^2-13y=-36;$$

whence $$y=9 \text{ or } 4.$$

But, since $x^2=y$, $$x=\pm\sqrt{y}.$$

Therefore $$x=\pm\sqrt{9} \text{ or } \pm\sqrt{4}.$$

Thus x has four values, viz., $+3$, -3, $+2$, -2.

To verify these values:

1st value, $(+3)^4-13(+3)^2=-36$, i. e., $81-117=-36$.

2d value, $(-3)^4-13(-3)^2=-36$, i. e., $81-117=-36$.

3d value, $(+2)^4-13(+2)^2=-36$, i. e., $16-\ 52=-36$.

4th value, $(-2)^4-13(-2)^2=-36$, i. e., $16-\ 52=-36$.

Ex. 2. Solve the equation $x^6-35x^3=-216$.

Assuming $x^3=y$, the above becomes

$$y^2-35y=-216;$$

whence $$y=27 \text{ or } 8.$$

Hence $$x=\sqrt[3]{y}=3 \text{ or } 2.$$

This equation has four other roots which can not be determined by this process.

Ex. 3. Solve the equation $x+4\sqrt{x}=21$.

Assuming $\sqrt{x}=y$, we have

$$y^2+4y=21;$$

whence $y=3$ or -7.
Therefore $x=9$ or 49.

Although the square root of 9 is generally ambiguous, and may be either $+3$ or -3, still, in verifying the preceding values, $\sqrt{x}$ can not be taken equal to -3, because 9 was obtained by multiplying $+3$ by itself. For a like reason, $\sqrt{x}$ can not be taken equal to $+7$. A similar remark is applicable to several of the following examples.

263. The same method of solution may often be extended to equations in which *any algebraic expression occurs with two exponents, one of which is double the other.*

Ex. 4. Solve the equation $(x^2+x)^2-26(x^2+x)=-120$.

Assuming $x^2+x=y$, this equation becomes

$$y^2-26y=-120;$$

whence $y=20$ or 6.

We have now the two equations,

$$x^2+x=20, \text{ and } x^2+x=6,$$

the first of which gives $x=-5$ or $+4$,
and the second gives $x=-3$ or $+2$.

Thus the equation has four roots,

$$-5, +4, -3, +2,$$

and any one of these four values will satisfy the given equation.

Ex. 5. Solve the equation $\sqrt{x+12}+\sqrt[4]{x+12}=6$.

We find $x+12=16$ or 81.
Hence $x=4$ or 69.

Ex. 6. Solve the equation $2x^2+\sqrt{2x^2+1}=11$.

This equation may be written

$$2x^2+1+\sqrt{2x^2+1}=12.$$

Hence $2x^2+1=9$ or 16;
therefore $x=+2, -2, +\sqrt{7\tfrac{1}{2}}, -\sqrt{7\tfrac{1}{2}}$.

264. *Equations of the Fourth Degree.*—An equation of the fourth degree may often be reduced to an equation containing the first and second powers of some compound quantity, with known coefficients, in the following manner: Transpose all the terms to the first member; then extract the square root to two terms, and see if the remainder (with or without the absolute term) is a multiple of the root already obtained.

Ex. 7. Solve the equation $x^4-12x^3+44x^2-48x=9009$.

We may proceed as follows:

$$\begin{array}{r|l} & x^4-12x^3+44x^2-48x-9009=0\,(x^2-6x \\ & x^4 \\ 2x^2-6x\,) & -12x^3+44x^2 \\ & -12x^3+36x^2 \\ \hline & \qquad 8x^2-48x-9009, \end{array}$$

or $$8(x^2-6x)-9009.$$

Hence the given equation may be expressed as follows:

$$(x^2-6x)^2+8(x^2-6x)=9009.$$

Ans. $x=13$ or -7, or $3\pm3\sqrt{-10}$.

Ex. 8. Solve the equation $x^4-2x^3+x=132$.

Ans. $x=4$ or -3, or $\frac{1}{2}\pm\frac{1}{2}\sqrt{-43}$.

Ex. 9. Solve the equation $x^4+4x^2=12$.

Ans. $x=\pm\sqrt{2}$ or $\pm\sqrt{-6}$.

Ex. 10. Solve the equation $x^6-8x^3=513$.

Ans. $x=3$ or $-\sqrt[3]{19}$.

Ex. 11. Solve the equation $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$.

Ans. $x=243$ or $-\sqrt[3]{28^5}$.

Ex. 12. Solve the equation $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$.

Ans. $x=\frac{1}{2}\sqrt[3]{2}$.

Ex. 13. Solve the equation $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$.

Ans. $x=\frac{1}{8}$ or -8.

Ex. 14. Solve the equation $\frac{1}{2}x-\frac{1}{3}\sqrt{x}=22\frac{1}{6}$.

Ans. $x=49$ or $\dfrac{361}{9}$.

Ex. 15. Solve the equation $\sqrt{10+x}-\sqrt[4]{10+x}=2$.

Ans. $x=6$ or -9.

Ex. 16. Solve the equation $x^6+20x^3-10=59$.

Ans. $x=\sqrt[3]{3}$ or $\sqrt[3]{-23}$.

Ex. 17. Solve the equation $3x^{2n}-2x^n+3=11$.

Ans. $x=\sqrt[n]{2}$ or $\sqrt[n]{-\frac{4}{3}}$.

Ex. 18. Solve the equation $\sqrt{1+x-x^2}-2(1+x-x^2)=\frac{1}{9}$.

Ans. $x=\frac{1}{2}\pm\frac{1}{6}\sqrt{41}$ or $\frac{1}{2}\pm\frac{1}{3}\sqrt{11}$.

Ex. 19. Solve the equation $\sqrt[4]{x}+\sqrt{x}=20$.

Ans. $x=256$ or 625.

Ex. 20. Solve the equation $x^4-4x^3+7x^2-6x=18$.

Ans. $x=3$ or -1, or $1\pm\sqrt{-5}$.

Ex. 21. Solve the equation $x^2+5x+4=5\sqrt{x^2+5x+28}$.

Ans. $x=4$ or -9, or $-\frac{5}{2}\pm\frac{1}{2}\sqrt{-51}$.

Ex. 22. Solve the equation $x^2+3=2\sqrt{x^2-2x+2}+2x$.

Ans. $x=1$.

Ex. 23. Solve the equation $(x+\sqrt{x})^4-(x+\sqrt{x})^2=20592$.

Ans. $x=9$ or 16.

Ex. 24. Solve the equation $x+\sqrt{25+x}=157$.

Ans. $x=144$ or 171.

Ex. 25. Solve the equation $\sqrt{x-1}=x-1$.

Ans. $x=1$ or 2.

265. We have seen that every equation of the second degree has two *roots;* that is, there are two quantities which, when substituted for x in the original equation, will render the two members identical. In like manner, we shall find that every equation of the third degree has *three roots*, an equation of the fourth degree has *four roots*, and, in general, *an equation of the* mth *degree has* m *roots.*

Before determining the degree of an equation, it should be freed from fractions, from negative exponents, and from the radical signs which affect its unknown quantities. Several of the preceding examples are thus found to furnish equations of the fourth degree, while others furnish equations of the second degree.

The above method of solving the equation $x^{2n}+px^n=q$ will not always give us *all* of the roots, and we must have recourse to different processes to discover the remaining roots. The subject will be more fully treated in Chapter XXI.

Problems producing Equations of the Second Degree.

Prob. 1. It is required to find two numbers such that their difference shall be 8 and their product 240.

Let $x=$the least number.

Then will $x+8=$the greater.

And by the question, $x(x+8)=x^2+8x=240$.

Therefore $\quad x=12$, the less number,

$\quad\quad x+8=20$, the greater.

Proof. $20-12=8$, the first condition.

$\quad 20\times12=240$, the second condition.

Prob. 2. The Receiving Reservoir at Yorkville is a rectangle, 60 rods longer than it is broad, and its area is 5500 square rods. Required its length and breadth.

Prob. 3. What two numbers are those whose difference is $2a$, and product b?

Ans. $a\pm\sqrt{a^2+b}$, and $-a\pm\sqrt{a^2+b}$.

Prob. 4. It is required to divide the number 60 into two such parts that their product shall be 864.

Let $x=$one of the parts.

Then will $60-x=$the other part.

And by the question, $x(60-x)=60x-x^2=864$.

The parts are 36 and 24, *Ans.*

Prob. 5. In a parcel which contains 52 coins of silver and copper, each silver coin is worth as many cents as there are copper coins, and each copper coin is worth as many cents as there are silver coins, and the whole are worth two dollars. How many are there of each?

Prob. 6. What two numbers are those whose sum is $2a$ and product b?

Ans. $a+\sqrt{a^2-b}$ and $a-\sqrt{a^2-b}$.

Prob. 7. There is a number consisting of two digits whose sum is 10, and the sum of their squares is 58. Required the number.

Let x=the first digit.

Then will $10-x$=the second digit.

And $x^2+(10-x)^2=2x^2-20x+100=58$;

that is, $x^2-10x=-21$,

$x^2-10x+25=4$,

$x=5\pm2=7$ or 3.

Hence the number is 73 or 37.

The two values of x are the required digits whose sum is 10. It will be observed that we put x to represent the first digit, whereas we find it may equal the second as well as the first. The reason is, that we have here imposed a condition which does not enter into the equation. If x represent *either* of the required digits, then $10-x$ will represent the *other*, and hence the values of x found by solving the equation should give both digits. Beginners are very apt thus, in the statement of a problem, to impose conditions which do not appear in the equation.

The preceding example, and all others of the same class, may be solved without completing the square. Thus,

Let x represent the half difference of the two digits.

Then, according to the principle on page 89, $5+x$ will represent the greater of the two digits, and $5-x$ the less.

The square of $5+x$ is $25+10x+x^2$,

" $5-x$ " $25-10x+x^2$.

The sum is $50+2x^2$, which, according to the problem, $=58$.

Hence $2x^2=8$,

or $x^2=4$,

and $x=\pm2$.

Therefore, $5+x=7$, the greater digit,

and $5-x=3$, the less digit.

Prob. 8. Find two numbers such that the product of their sum and difference may be 5, and the product of the sum of their squares and the difference of their squares may be 65.

Prob. 9. Find two numbers such that the product of their sum and difference may be a, and the product of the sum of their squares and the difference of their squares may be ma.

Ans. $\sqrt{\frac{m+a}{2}}$; $\sqrt{\frac{m-a}{2}}$.

Prob. 10. A laborer dug two trenches, whose united length was 26 yards, for 356 shillings, and the digging of each of them cost as many shillings per yard as there were yards in its length. What was the length of each?

Ans. 10, or 16 yards.

Prob. 11. What two numbers are those whose sum is $2a$, and the sum of their squares is $2b$?

Ans. $a+\sqrt{b-a^2}$, and $a-\sqrt{b-a^2}$.

Prob. 12. A farmer bought a number of sheep for 80 dollars, and if he had bought four more for the same money, he would have paid one dollar less for each. How many did he buy?

Let x represent the number of sheep.

Then will $\frac{80}{x}$ be the price of each.

And $\frac{80}{x+4}$ would be the price of each if he had bought four more for the same money.

But by the question we have

$$\frac{80}{x}=\frac{80}{x+4}+1.$$

Solving this equation, we obtain $x=16$, *Ans.*

Prob. 13. A person bought a number of articles for a dollars. If he had bought $2b$ more for the same money, he would have paid c dollars less for each. How many did he buy?

Ans. $-b\pm\sqrt{\frac{2ab+b^2c}{c}}$.

Prob. 14. It is required to find three numbers such that the product of the first and second may be 15, the product of the first and third 21, and the sum of the squares of the second and third 74. *Ans.* 3, 5, and 7.

Prob. 15. It is required to find three numbers such that the product of the first and second may be a, the product of the first and third b, and the sum of the squares of the second and third c.

$$Ans.\ \sqrt{\frac{a^2+b^2}{c}};\ a\sqrt{\frac{c}{a^2+b^2}};\ b\sqrt{\frac{c}{a^2+b^2}}.$$

Prob. 16. The sum of two numbers is 16, and the sum of their cubes 1072. What are the numbers? *Ans.* 7 and 9.

Prob. 17. The sum of two numbers is $2a$, and the sum of their cubes is $2b$. What are the numbers?

$$Ans.\ a+\sqrt{\frac{b-a^3}{3a}} \text{ and } a-\sqrt{\frac{b-a^3}{3a}}.$$

Prob. 18. Two magnets, whose powers of attraction are as 4 to 9, are placed at a distance of 20 inches from each other. It is required to find, on the line which joins their centres, the point where a needle would be equally attracted by both, admitting that the intensity of magnetic attraction varies inversely as the square of the distance.

Ans. { 8 inches from the weakest magnet, or -40 inches from the weakest magnet.

Prob. 19. Two magnets, whose powers are as m to n, are placed at a distance of a feet from each other. It is required to find, on the line which joins their centres, the point which is equally attracted by both.

Ans. { The distance from the magnet m is $\dfrac{a\sqrt{m}}{\sqrt{m}\pm\sqrt{n}}$.
The distance from the magnet n is $\dfrac{\pm a\sqrt{n}}{\sqrt{m}\pm\sqrt{n}}$.

Prob. 20. A set out from C toward D, and traveled 6 miles an hour. After he had gone 45 miles, B set out from D toward C, and went every hour $\frac{1}{20}$ of the entire distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance between the places C and D. *Ans.* Either 100 miles, or 180 miles.

Prob. 21. A set out from C toward D, and traveled a miles per hour. After he had gone b miles, B set out from D toward C, and went every hour $\frac{1}{n}$th of the entire distance; and after he had traveled as many hours as he went miles in one hour, he met A. Required the distance between the places C and D.

$$Ans.\ n\left(\frac{n-a}{2}\pm\sqrt{\left(\frac{n-a}{2}\right)^2-b}\right).$$

Prob. 22. By selling my horse for 24 dollars, I lose as much per cent. as the horse cost me. What was the first cost of the horse? *Ans.* 40 or 60 dollars.

Prob. 23. A fruit-dealer receives an order to buy 18 melons provided they can be bought at 18 cents a piece; but if they should be dearer or cheaper than 18 cents, he is to buy as many less or more than 18 as each costs more or less than 18 cents. He paid in all \$3.15. How many melons did he buy? *Ans.* Either 15 or 21.

Prob. 24. A line of given length (a) is bisected and produced; find the length of the produced part, so that the rectangle contained by half the line, and the line made up of the half and the produced part, may be equal to the square on the produced part. $x^2 = (\frac{a}{2})(\frac{a}{2}+x) = \frac{a^2}{4} + \frac{ax}{2}$ ∴ $x^2 - \frac{ax}{2} = \frac{a^2}{4}$ ... $\frac{a^2}{16} = \frac{5a^2}{16}$ ∴ $x - \frac{a}{4} = \frac{a}{4}\sqrt{5}$ ∴ *Ans.* $\frac{a}{4}(1+\sqrt{5})$.

Equations of the Second Degree containing Two Unknown Quantities.

266. An equation containing two unknown quantities is said to be of *the second degree when the highest sum of the exponents of the unknown quantities in any term is two.* Thus

$$x^2+y^2=13, \qquad (1.)$$

and

$$x+xy+y=11, \qquad (2.)$$

are equations of the second degree.

267. The solution of two equations of the second degree containing two unknown quantities generally involves the solution of an equation of the fourth degree containing one unknown quantity. Thus, from equation (2), we find

$$y=\frac{11-x}{x+1}.$$

Substituting this value for y in equation (1) and reducing, we have

$$x^4+2x^3-11x^2-48x=-108,$$

an equation which can not be solved by the preceding methods. Yet there are particular cases in which simultaneous equations of a degree higher than the first may be solved by the rules for quadratic equations. The following are the principal cases of this kind:

268. 1*st. When one of the equations is of the first degree and the other of the second.*—We find an expression for the value of one of the unknown quantities in the former equation, and substitute this value for its equal in the other equation.

Ex. 1. Given $\left\{\begin{matrix} x^2+3xy-y^2=23 \\ x+2y=7 \end{matrix}\right\}$ to find x and y.

From the second equation we find

$$x=7-2y.$$

Substituting this value for x in the first equation, we have

$$49-28y+4y^2+21y-6y^2-y^2=23,$$

which may be solved in the usual manner.

Ans. $\begin{cases} x=3 \text{ or } \frac{47}{3}. \\ y=2 \text{ or } -\frac{13}{3}. \end{cases}$

Ex. 2. Given $\left\{\begin{matrix} 2x^2+xy-5y^2=20 \\ 2x-3y=1 \end{matrix}\right\}$ to find x and y.

Ans. $\begin{cases} x=5 \text{ or } -\frac{37}{4}. \\ y=3 \text{ or } -\frac{13}{2}. \end{cases}$

Ex. 3. Given $\left\{\begin{matrix} \frac{10x+y}{3}=xy \\ 9y-9x=18 \end{matrix}\right\}$ to find x and y.

Ans. $\begin{cases} x=2 \text{ or } -\frac{1}{3}. \\ y=4 \text{ or } \frac{5}{3}. \end{cases}$

For equations of this class there are in general two sets of values of x and y.

269. 2*d. When both of the equations are of the second degree, and homogeneous.*—Substitute for one of the unknown quantities the product of the other by a third unknown quantity.

Ex. 4. Given $\begin{cases} x^2+xy=12 \\ xy-2y^2=1 \end{cases}$ to find x and y.

If we assume $x=vy$, we shall have

$$v^2y^2+vy^2=12, \text{ whence } y^2=\frac{12}{v^2+v};$$

$$vy^2-2y^2=1, \quad \text{"} \quad y^2=\frac{1}{v-2}.$$

Therefore $$\frac{12}{v^2+v}=\frac{1}{v-2}.$$

From which we obtain $v=8$ or 3.

Substituting either of these values in one of the preceding expressions for y^2, we shall obtain the values of y; and since $x=vy$, we may easily obtain the values of x.

$$Ans. \begin{cases} x=\pm 3 \text{ or } \pm\frac{8}{\sqrt{6}}. \\ y=\pm 1 \text{ or } \pm\frac{1}{\sqrt{6}}. \end{cases}$$

Ex. 5. Given $\begin{cases} x^2+xy=77 \\ xy-y^2=12 \end{cases}$ to find x and y.

Assuming $x=vy$, we find $v=\frac{7}{4}$ or $\frac{11}{3}$.

$$Ans. \begin{cases} x=\pm 7 \text{ or } \pm\frac{11}{\sqrt{2}}. \\ y=\pm 4 \text{ or } \pm\frac{3}{\sqrt{2}}. \end{cases}$$

Ex. 6. Given $\begin{cases} 2x^2+3xy+y^2=20 \\ 5x^2+4y^2=41 \end{cases}$ to find x and y.

Assuming $x=vy$, we find $v=\frac{1}{3}$ or $\frac{13}{2}$.

$$Ans. \begin{cases} x=\pm 1 \text{ or } \pm\frac{13}{\sqrt{21}}. \\ y=\pm 3 \text{ or } \pm\frac{2}{\sqrt{21}}. \end{cases}$$

For equations of this class there are *in general* four sets of values of x and y. It should be borne in mind that to any one of the four values of x there corresponds only one of the four values of y. Thus, when x in the 6th example is $+1$, y must be $+3$, and can not be one of the other three values given above.

270. 3*d. When the unknown quantities enter each equation symmetrically.*—Substitute for the unknown quantities the sum and difference of two other quantities, or the sum and product of two other quantities.

Ex. 7. Given $\left\{\begin{array}{l}\frac{x^2}{y}+\frac{y^2}{x}=18 \\ x+y=12\end{array}\right\}$ to find x and y.

Let us assume $$x=z+v,$$ $$y=z-v.$$

Then $$x+y=2z=12 \text{ or } z=6.$$

That is, $$x=6+v \text{ and } y=6-v.$$

But from the first equation we find

$$x^3+y^3=18xy.$$

Substituting the preceding values of x and y in this equation, and reducing, we have

$$432+36v^2=648-18v^2,$$

whence $$v=\pm 2.$$

Therefore $$x=4 \text{ or } 8, \text{ and } y=8 \text{ or } 4.$$

Ex. 8. Given $\left\{\begin{array}{l}x^5+y^5=3368 \\ x+y=8\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=3 \text{ or } 5. \\ y=5 \text{ or } 3.\end{array}\right.$

Ex. 9. Given $\left\{\begin{array}{l}x^3+y^3=341 \\ x^2y+xy^2=330\end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l}x=5 \text{ or } 6. \\ y=6 \text{ or } 5.\end{array}\right.$

271. 4*th. When the same algebraic expression is involved to different powers*, it is sometimes best to regard this expression as the unknown quantity.

Ex. 10. Given $\left\{\begin{array}{c} x^2+2xy+y^2+2x=120-2y \\ xy-y^2=8 \end{array}\right\}$ to find x and y.

The first equation may be written

$$(x+y)^2+2(x+y)=120.$$

Regarding $x+y$ as a single quantity, we find its value to be either 10 or -12.

Proceeding now as in *Art.* 268, we find

$$x=6 \text{ or } 9, \text{ or } -9\mp\sqrt{5};$$
$$y=4 \text{ or } 1, \text{ or } -3\pm\sqrt{5}.$$

Ex. 11. Given $\left\{\begin{array}{c} 4xy=96-x^2y^2 \\ x+y=6 \end{array}\right\}$ to find x and y.

Regarding xy as the unknown quantity, its value from the first equation is found to be either

$$8 \text{ or } -12.$$

Ans. $\left\{\begin{array}{l} x=2 \text{ or } 4, \text{ or } 3\pm\sqrt{21}. \\ y=4 \text{ or } 2, \text{ or } 3\mp\sqrt{21}. \end{array}\right.$

Ex. 12. Given $\left\{\begin{array}{c} \frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9} \\ x-y=2 \end{array}\right\}$ to find x and y.

Regarding $\frac{x}{y}$ as the unknown quantity, we find its value to be either $\frac{5}{3}$ or $-\frac{17}{3}$.

Ans. $\left\{\begin{array}{l} x=5 \text{ or } \frac{17}{10}. \\ y=3 \text{ or } -\frac{3}{10}. \end{array}\right.$

For several of these examples there are other roots, some of which can not be obtained by the processes heretofore explained. The roots of two simultaneous equations are sometimes infinite, as in the 8th and 9th examples, where the equations may be satisfied by $x=\pm\infty$, $y=\mp\infty$, since two quantities that are infinitely great may differ by a finite quantity.

Solve the following groups of simultaneous equations:

Ex. 13. $\left\{\begin{array}{c} x^2-2xy-y^2=1 \\ x+y=2 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=\pm\sqrt{\frac{5}{2}}. \\ y=2\mp\sqrt{\frac{5}{2}}. \end{array}\right.$

Ex. 14. $\left\{\begin{array}{c} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{a}{x}+\frac{b}{y}=4 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=\frac{a}{2}. \\ y=\frac{b}{2}. \end{array}\right.$

Ex. 15. $\begin{cases} x^2+y^2+xy=84 \\ x+y+\sqrt{xy}=14 \end{cases}$ *Ans.* $\begin{cases} x=8 \text{ or } 2. \\ y=2 \text{ or } 8. \end{cases}$

Ex. 16. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{5} \\ \frac{10}{xy}=\frac{1}{18} \end{cases}$ *Ans.* $\begin{cases} x=6 \text{ or } 30. \\ y=30 \text{ or } 6. \end{cases}$

Ex. 17. $\begin{cases} (x^2+y^2)x^2y^2=468 \\ (x+y)xy=30 \end{cases}$ *Ans.* $\begin{cases} x=2. \\ y=3. \end{cases}$

Ex. 18. $\begin{cases} xy=a \\ x^2+y^2=b \end{cases}$ *Ans.* $\begin{cases} x=\pm\frac{1}{2}\sqrt{b+2a}\pm\frac{1}{2}\sqrt{b-2a}. \\ y=\mp\frac{1}{2}\sqrt{b+2a}\mp\frac{1}{2}\sqrt{b-2a}. \end{cases}$

Ex. 19. $\begin{cases} x+y=72 \\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=6 \end{cases}$ *Ans.* $\begin{cases} x=8 \text{ or } 64. \\ y=64 \text{ or } 8. \end{cases}$

Ex. 20. $\begin{cases} x^2y+y^2x=20 \\ \frac{1}{x}+\frac{1}{y}=\frac{5}{4} \end{cases}$ *Ans.* $\begin{cases} x=1 \text{ or } 4. \\ y=4 \text{ or } 1. \end{cases}$

Ex. 21. $\begin{cases} x^2+y^2=8 \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{2} \end{cases}$ *Ans.* $\begin{cases} x=\pm 2. \\ y=\pm 2. \end{cases}$

Ex. 22. $\begin{cases} x^5-y^5=3093 \\ x-y=3 \end{cases}$ *Ans.* $\begin{cases} x=5 \text{ or } -2. \\ y=2 \text{ or } -5. \end{cases}$

Ex. 23. $\begin{cases} x^4+x^2y^2+y^4=931 \\ x^2+xy+y^2=49 \end{cases}$ *Ans.* $\begin{cases} x=\pm 5 \text{ or } \pm 3. \\ y=\pm 3 \text{ or } \pm 5. \end{cases}$

Ex. 24. $\begin{cases} (7+x)(6+y)=80 \\ x+y=5 \end{cases}$ *Ans.* $\begin{cases} x=1 \text{ or } 3. \\ y=4 \text{ or } 2. \end{cases}$

Note. Put $7+x=z$, $6+y=v$.

PROBLEMS.

1. Divide the number 100 into two such parts that the sum of their square roots may be 14. *Ans.* 64 and 36.

2. Divide the number a into two such parts that the sum of their square roots may be b.

Ans. $\frac{a}{2}\pm\frac{b}{2}\sqrt{2a-b^2}$.

3. The sum of two numbers is 8, and the sum of their fourth powers is 706. What are the numbers? *Ans.* 3 and 5.

4. The sum of two numbers is $2a$, and the sum of their fourth powers is $2b$. What are the numbers?

Ans. $a \pm \sqrt{-3a^2+\sqrt{8a^4+b}}$.

5. The sum of two numbers is 6, and the sum of their fifth powers is 1056. What are the numbers? *Ans.* 2 and 4.

6. The sum of two numbers is $2a$, and the sum of their fifth powers is b. What are the numbers?

Ans. $a \pm \sqrt{\sqrt{\frac{b}{10a}+\frac{4a^4}{5}}-a^2}$.

7. What two numbers are those whose product is 120; and if the greater be increased by 8 and the less by 5, the product of the two numbers thus obtained shall be 300?

Ans. 12 and 10, or 16 and 7.5.

8. What two numbers are those whose product is a; and if the greater be increased by b and the less by c, the product of the two numbers thus obtained shall be d?

Ans. $\frac{m}{2} \pm \sqrt{\frac{m^2}{4}-\frac{ab}{c}}$, and $\frac{a}{\frac{m}{2} \pm \sqrt{\frac{m^2}{4}-\frac{ab}{c}}}$,

where $m = \frac{d-a-bc}{c}$.

9. Find two numbers such that their sum, their product, and the difference of their squares may be all equal to one another.

Ans. $\frac{3}{2}+\sqrt{\frac{5}{4}}$, and $\frac{1}{2}+\sqrt{\frac{5}{4}}$,

that is, 2.618, and 1.618, nearly.

10. Divide the number 100 into two such parts that their product may be equal to the difference of their squares.

Ans. 38.197, and 61.803.

11. Divide the number a into two such parts that their product may be equal to the difference of their squares.

Ans. $\frac{3a \pm a\sqrt{5}}{2}$ and $\frac{-a \mp a\sqrt{5}}{2}$.

12. The sum of two numbers is a, and the sum of their reciprocals is b. Required the numbers.

Ans. $\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{a}{b}}$.

General Properties of Equations of the Second Degree.

272. *Every equation of the second degree containing but one unknown quantity has two roots, and only two.*

We have seen, *Art.* 250, that every equation of the second degree containing but one unknown quantity can be reduced to the form $x^2+px=q$. We have also found, *Art.* 257, that this equation has two roots, viz.,

$$x=-\frac{p}{2}+\sqrt{q+\frac{p^2}{4}}, \text{ and } -\frac{p}{2}-\sqrt{q+\frac{p^2}{4}}.$$

This equation can not have *more* than two roots; for, if possible, suppose it to have three roots, and represent these roots by x', x'', and x'''. Then, since a root of an equation is such a number as, substituted for the unknown quantity, will satisfy the equation, we must have

$$x'^2+px'=q, \qquad (1.)$$
$$x''^2+px''=q, \qquad (2.)$$
$$x'''^2+px'''=q. \qquad (3.)$$

Subtracting (2) from (1), we have

$$x'^2-x''^2+p(x'-x'')=0.$$

Dividing by $x'-x''$, we have

$$(x'+x'')+p=0. \qquad (4.)$$

In the same manner, we find

$$(x'+x''')+p=0. \qquad (5.)$$

Subtracting (5) from (4), we have

$$x''-x'''=0;$$

that is, the third supposed root is identical with the second; hence there can not be three different roots to a quadratic equation.

273. *The algebraic sum of the two roots is equal to the coefficient of the second term of the equation taken with the contrary sign.*

If we add together the two values of x in the general equation, the radical parts having opposite signs disappear, and we obtain

$$-\frac{p}{2}-\frac{p}{2}=-p.$$

Thus, in the equation $x^2-10x=-16$, the two roots are 8 and 2, whose sum is $+10$, the coefficient of x taken with the contrary sign.

If the two roots are equal numerically, but have opposite signs, their sum is zero, and the second term of the equation vanishes. Thus the two roots of the equation $x^2=16$ are $+4$ and -4, whose sum is zero. This equation may be written

$$x^2+0x=16.$$

274. *The product of the two roots is equal to the second member of the equation taken with the contrary sign.*

If we multiply together the two values of x (observing that the product of the sum and difference of two quantities is equal to the difference of their squares), we obtain

$$\frac{p^2}{4}-\left(q+\frac{p^2}{4}\right)=-q.$$

Thus, in the equation $x^2-10x=-16$, the product of the two roots 8 and 2 is $+16$, which is equal to the second member of the equation taken with the contrary sign.

275. The last two principles enable us to *form an equation* whose roots shall be any given quantities.

Ex. 1. Find the equation whose roots are 3 and 5.

According to *Art.* 273, the coefficient of the second term of the equation must be -8; and, according to *Art.* 274, the second member of the equation must be -15. Hence the equation is

$$x^2-8x=-15.$$

Ex. 2. Find the equation whose roots are -4 and -7.

Ex. 3. Find the equation whose roots are 5 and -9.

Ex. 4. Find the equation whose roots are -6 and $+11$.

Ex. 5. Find the equation whose roots are 1 and -2.

Ex. 6. Find the equation whose roots are $-\frac{1}{3}$ and $+\frac{1}{2}$.

Ex. 7. Find the equation whose roots are $-\frac{1}{3}$ and $+\frac{1}{4}$.

Ex. 8. Find the equation whose roots are $1 \pm \sqrt{5}$.

Ex. 9. Find the equation whose roots are $1 \pm \sqrt{-5}$.

276. *Every equation of the second degree whose roots are a and b, may be reduced to the form* $(x-a)(x-b)=0$.

Take the general equation

$$x^2+px=q,$$

and write it $\quad x^2+px-q=0.$

Then, by *Art.* 273, $\quad p=-(a+b)$;

and by *Art.* 274, $\quad q=-ab.$

Hence, by substitution,

$$x^2-(a+b)x+ab=0;$$

or, resolving into factors,

$$(x-a)(x-b)=0.$$

Thus the equation $x^2-10x=-16$, whose roots are 8 and 2, may be resolved into the factors $x-8=0$ and $x-2=0$.

It is also obvious that if a is a root of an equation of the second degree, the equation must be divisible by $x-a$. Thus the preceding equation is divisible by $x-8$, giving the quotient $x-2$.

Ex. 1. The roots of the equation $x^2+6x+8=0$ are -2 and -4. Resolve the first member into its factors.

Ex. 2. The roots of the equation $x^2+6x-27=0$ are $+3$ and -9. Resolve the first member into its factors.

Ex. 3. The roots of the equation $x^2-2x-24=0$ are $+6$ and -4. Resolve the first member into its factors.

Ex. 4. Resolve the equation $x^2+73x+780=0$ into simple factors. *Ans.* $(x+60)(x+13)=0$.

Ex. 5. Resolve the equation $x^2-88x+1612=0$ into simple factors. *Ans.* $(x-62)(x-26)=0$.

Ex. 6. Resolve the equation $2x^2+x-6=0$ into simple factors. *Ans.* $2(x+2)(x-\frac{3}{2})=0$.

Ex. 7. Resolve the equation $3x^2-10x-25=0$ into simple factors. *Ans.* $3(x-5)(x+\frac{5}{3})=0$.

Discussion of the General Equation of the Second Degree.

277. In the general equation of the second degree $x^2+px=q$, the coefficient of x, as well as the absolute term, may be either positive or negative. We may therefore have the four following forms:

First form, $x^2+px=q$.
Second form, $x^2-px=q$.
Third form, $x^2+px=-q$.
Fourth form, $x^2-px=-q$.

From these equations we obtain

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

$$x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}.$$

$$x=+\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}.$$

We will now consider what conditions will render these roots positive or negative, equal or unequal, real or imaginary.

278. *Positive and negative roots.*

Since $\frac{p^2}{4}+q$ is greater than $\frac{p^2}{4}$,

$$\sqrt{\frac{p^2}{4}+q} \text{ must be greater than } \frac{p}{2}.$$

For the same reason, $\sqrt{\frac{p^2}{4}-q}$ must be less than $\frac{p}{2}$.

Therefore, in the first and second forms, the sign of the roots will correspond to the sign of the radicals; but in the third and fourth forms the sign of the roots will correspond to the sign of the rational parts. Hence, *in the first form, one root is positive and the other negative, and the negative root is numerically the greatest;* as in the equation $x^2+x=6$, whose roots are $+2$ and -3.

In the second form one root is positive and the other negative, and the positive root is numerically the greatest, as in the equation $x^2-x=6$, whose roots are -2 and $+3$.

In the third form both roots are negative, as in the equation $x^2+5x=-6$, whose roots are -2 and -3.

In the fourth form both roots are positive, as in the equation $x^2-5x=-6$, whose roots are $+2$ and $+3$.

279. *Equal and unequal roots.*

In the first and second forms the two roots are always unequal; but *in the third and fourth forms*, when q is numerically equal to $\frac{p^2}{4}$, the radical part of both values of x becomes zero, and *the two roots are then said to be equal.* In this case

the third form gives $x=-\frac{p}{2}\pm0=-\frac{p}{2}$,

and the fourth form gives

$$x=+\frac{p}{2}\pm0=+\frac{p}{2}.$$

Thus, in the equation $x^2+6x=-9$, the two roots are -3 and -3. We say that in this case the equation has two roots, because it is the product of the two factors $x+3=0$ and $x+3=0$.

So, also, in the equation $x^2-6x=-9$, the two roots are $+3$ and $+3$.

280. *Real and imaginary roots.*

Since $\frac{p^2}{4}$, being a square, is positive for all real values of p, it follows that the expression $\frac{p^2}{4}+q$ can only be rendered negative by the sign of q; that is, the quantity under the radical sign can only be negative when q is negative and numerically greater than $\frac{p^2}{4}$. Hence, *in the first and second forms, both roots are always real;* but *in the third and fourth forms both roots are imaginary when q is numerically greater than $\frac{p^2}{4}$.*

Thus, in the equation $x^2+4x=-6$, the two roots are

$$-2\pm\sqrt{-2};$$

and in the equation $x^2-4x=-6$, the two roots are

$$+2\pm\sqrt{-2}.$$

It will be observed that when one of the roots is imaginary, the other is imaginary also.

281. *Imaginary roots indicate impossible conditions in the proposed question which furnished the equation.*

The demonstration of this principle depends upon the following proposition: *the greatest product which can be obtained by dividing a number into two parts and multiplying them together is the square of half that number.*

Let p represent the given number, and d the difference of the parts.

Then, from page 89, $\frac{p}{2}+\frac{d}{2}=$ the greater part,

and $\frac{p}{2}-\frac{d}{2}=$ the less part,

and $\frac{p^2}{4}-\frac{d^2}{4}=$ the product of the parts.

Now, since p is a given quantity, it is plain that the product will be the greatest possible when $d=0$; that is, the greatest product is the square of $\frac{p}{2}$, half the given number.

For example, let 10 be the number to be divided.

We have
$10=1+9$; and $9\times1=9$.
$10=2+8$; and $8\times2=16$.
$10=3+7$; and $7\times3=21$.
$10=4+6$; and $6\times4=24$.
$10=5+5$; and $5\times5=25$.

Thus we see that the smaller the difference of the two parts, the greater is their product; and this product is greatest when the two parts are equal.

Now, in the equation $x^2-px=-q$, p is the sum of the two roots, and q is their product. Therefore q can never be greater than $\frac{p^2}{4}$.

If, then, any problem furnishes an equation in which q is negative, and numerically greater than $\frac{p^2}{4}$, we infer that the conditions of the question are incompatible with each other.

Suppose it is required to divide 6 into two parts such that their product shall be 10.

Let x represent one of the parts, and $6-x$ the other part.

Then, by the conditions,

$$x(6-x)=10;$$

whence $$x^2-6x=-10,$$

and $$x=3\pm\sqrt{-1}.$$

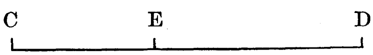
The imaginary value of x indicates that it is impossible to find two numbers whose sum is 6 and product 10. From the preceding proposition, it appears that 9 is the greatest product which can be obtained by dividing 6 into two parts and multiplying them together.

Discussion of Particular Problems.

282. In discussing particular problems which involve equations of the second degree, we meet with all the different cases which are presented by equations of the first degree, and some peculiarities besides. All the different cases enumerated in Chapter X. are presented by Prob. 19, page 195, when we make different suppositions upon the values of a, m, and n; but we need not dwell upon them.

The peculiarities exhibited by equations of the second degree are double values of x, and imaginary values.

283. *Double Values of the Unknown Quantity.*—We have seen that every equation of the second degree has two roots. Sometimes both of these values are applicable to the problem which furnishes the equation. Thus, in Prob. 20, page 195, we obtain either 100 or 180 miles for the distance between the places C and D.



Let E represent the position of A when B sets out on his journey. Then, if we suppose CD equals 100 miles, ED will equal 55 miles, of which A will travel 30 miles (being 6 miles an hour for 5 hours), and B will travel 25 miles (being 5 miles an hour for 5 hours).

If we suppose CD equals 180 miles, ED will equal 135 miles, of which A will travel 54 miles (being 6 miles an hour for 9 hours), and B will travel 81 miles (being 9 miles an hour for 9 hours).

This problem, therefore, admits of two positive answers, both equally applicable to the question. Problems 22 and 23, page 196, are of the same kind.

In Problem 18, page 195, one of the values of x is positive and the other negative.

Let the weaker magnet be placed at A, and the stronger at B; then C will represent the position of a needle equally attracted by both magnets. According to the first value, the distance AC=8 inches, and CB=12 inches. Now, at the distance of 8 inches, the attraction of the weaker magnet will be represented by $\frac{4}{8^2}$; and at the distance of 12 inches, the attraction of the other magnet will be represented by $\frac{9}{12^2}$, and these two powers are equal; for

$$\frac{4}{8^2}=\frac{9}{12^2}.$$

But there is another point, C′, which equally satisfies the conditions of the question; and this point is 40 inches to the left of A, and therefore 60 inches to the left of B; for

$$\frac{4}{40^2}=\frac{9}{60^2}.$$

284. *Imaginary Values of the Unknown Quantity.*—We have seen that an imaginary root indicates impossible conditions in the proposed question which furnished the equation. In several of the preceding problems the values of x become imaginary in particular cases.

When will the values of x in Prob. 6, page 192, be imaginary? *Ans.* When $b > a^2$.

What is the impossibility involved in this supposition?

Ans. It is impossible that the product of two numbers can be greater than the square of half their sum.

When will the values of x in Prob. 11, page 194, be imaginary? *Ans.* When $a^2 > b$; or $(2a)^2 > 4b$.

What is the impossibility involved in this supposition?

Ans. The square of the sum of two numbers can not be greater than twice the sum of their squares.

When will the values of x in Prob. 17, page 195, be imaginary? *Ans.* When $a^3 > b$; or $(2a)^3 > 8b$.

What is the impossibility of this supposition?

Ans. The cube of the sum of two numbers can not be greater than four times the sum of their cubes.

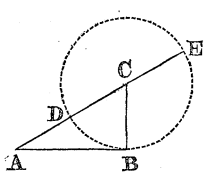
When will the values of x in Prob. 4, page 180, be imaginary, and what is the impossibility of this supposition?

285. *Geometrical Construction of Equations of the Second Degree.*—The roots of an equation of the second degree may be represented by a simple geometrical figure. This may be done for each of the four forms:

First form.—The first form gives for x the two values

$$x = -\frac{p}{2} + \sqrt{\frac{p^2}{4} + q}, \text{ and } x = -\frac{p}{2} - \sqrt{\frac{p^2}{4} + q}.$$

Draw the line AB, and make it equal to $\sqrt{q}$. From B draw BC perpendicular to AB, and make it equal to $\frac{p}{2}$. Join A and C; then AC will represent the value of $\sqrt{\frac{p^2}{4} + q}$. For $AC^2 = AB^2 + BC^2$ (Geom., Prop. 11, Bk. IV.).



With C as a centre, and CB as a radius, describe a circle cutting AC in D, and AC produced in E. For the first value of x the radical is positive, and is set off from A toward C; then $-\frac{p}{2}$ is set off from C to D; and AD, estimated from A to D, represents the first value of x.

For the second value of x we begin at E, and set off EC equal to $-\frac{p}{2}$; we then set off the minus radical from C to A; and EA, estimated from E to A, represents the second value of x.

Second form.—The second form gives for x the two values

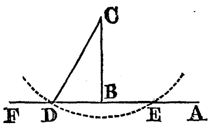
$$x=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q}, \text{ and } x=\frac{p}{2}-\sqrt{\frac{p^2}{4}+q}.$$

The first value of x is represented by AE estimated from A to E. The second value is $+$DC$-$CA, the latter being estimated from C to A. Hence the second value is represented by DA estimated from D to A.

Third form.—The third form gives for x the two values

$$x=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}, \text{ and } x=-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}.$$

Draw an indefinite line FA, and from any point, as A, set

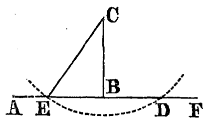


off a distance AB$=-\frac{p}{2}$. We set off its value to the left, because $\frac{p}{2}$ is negative. At B draw BC perpendicular to FA, and make it equal to $\sqrt{q}$. With C as a centre, and a radius equal to $\frac{p}{2}$, describe an arc of a circle cutting FA in D and E. Now the value of $\sqrt{\frac{p^2}{4}-q}$ will be BD or BE. The first value of x will be represented by $-$AB$+$BE, which is equal to $-$AE. The second will be represented by $-$AB$-$BD, which is equal to $-$AD; so that both of the roots are negative, and are estimated in the same direction, from A toward the left.

Fourth form.—The fourth form gives for x the two values

$$x=\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}, \text{ and } x=\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}.$$

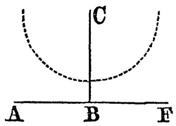
Construct the radical part of the value of x as in the last case. Then, since $\frac{p}{2}$ is positive, we set off its value AB from



A toward the right. To AB we add BD, which gives AD for the first value of x; and from AB we subtract BE, which leaves AE for the second value of x. Both values are positive, and are estimated in the same direction, from A toward the right.

Equal Roots.—If the radius CE be taken equal to CB, that is, if $\sqrt{q}$ is equal to $\frac{p}{2}$, the arc described with the centre C will be tangent to AF, the two points D and E will unite, and the two values of x become equal to each other. In this case the radical part of the value of x becomes zero.

Imaginary Roots.—If the radius of the circle described with the centre C be taken less than CB, it will not cut the line AF. In this case q is numerically greater than $\frac{p^2}{4}$, and the radical part of the value of x is imaginary.



CHAPTER XV.

RATIO AND PROPORTION.

286. *Ratio* is the relation which one quantity bears to another with respect to magnitude. Ratio is denoted by two points like the colon (:) placed between the quantities compared. Thus the ratio of a to b is written $a : b$.

The first quantity is called the *antecedent* of the ratio, and the second the *consequent.* The two quantities compared are called the *terms* of the ratio, and together they form a *couplet.* The quantities compared may be *polynomials;* nevertheless, each quantity is called one term of the ratio.

287. A ratio is *measured* by the fraction whose numerator is the antecedent and whose denominator is the consequent of the ratio. Thus the ratio of a to b is measured by $\frac{a}{b}$.

288. A *compound ratio* is the ratio arising from multiplying together the corresponding terms of two or more simple ratios. Thus the ratio of a to b compounded with the ratio of c to d becomes ac to bd.

The ratio compounded of the ratios 3 to 5 and 7 to 9 is 21 to 45.

289. The *duplicate ratio* of two quantities is the ratio of their squares. Thus the duplicate ratio of 2 to 3 is 4 to 9; the duplicate ratio of a to b is a^2 to b^2.

290. The *triplicate ratio* of two quantities is the ratio of their cubes. Thus the triplicate ratio of a to b is a^3 to b^3.

291. *If the terms of a ratio are both multiplied or both divided* *by the same quantity, the value of the ratio remains unchanged.* The ratio of a to b is represented by the fraction $\frac{a}{b}$, and the value of a fraction is not changed if we multiply or divide both numerator and denominator by the same quantity. Thus

$$\frac{a}{b}=\frac{ma}{mb},$$

or

$$a:b=ma:mb=\frac{a}{n}:\frac{b}{n}.$$

PROPORTION.

292. *Proportion is an equality of ratios.* Thus, if a, b, c, d are four quantities such that a when divided by b gives the same quotient as c when divided by d, these four quantities are called proportionals. This proportion may be written thus,

$$a:b::c:d,$$

or

$$a:b=c:d,$$

or

$$\frac{a}{b}=\frac{c}{d}.$$

In either case the proportion is read *a is to b as c is to d.*

293. The *terms* of a proportion are the four quantities which are compared. The first and fourth terms are called the *extremes*, the second and third the *means.* The first term is called the *first antecedent*, the second term the *first consequent*, the third term the *second antecedent*, and the fourth term the *second consequent.*

294. When the first of a series of quantities has the same ratio to the second that the second has to the third, or the third to the fourth, and so on, these quantities are said to be in *continued proportion*, and any one of them is a *mean proportional* between the two adjacent ones. Thus, if

$$a:b::b:c::c:d::d:e,$$

then a, b, c, d, and e are in continued proportion, and b is a mean proportional between a and c, c is a mean proportional between b and d, and so on.

295. *Alternation* is when antecedent is compared with antecedent and consequent with consequent. Thus, if

$$a:b::c:d,$$

then, by alternation, $a:c::b:d.$ See Art. 301.

296. *Inversion* is when antecedents are made consequents, and consequents are made antecedents. Thus, if

$$a:b::c:d,$$

then, inversely, $b:a::d:c.$ See Art. 302.

297. *Composition* is when the sum of antecedent and consequent is compared with either antecedent or consequent. Thus, if $a:b::c:d,$

then, by composition, $a+b:a::c+d:c,$

and $a+b:b::c+d:d.$ See Art. 304.

298. *Division* is when the difference of antecedent and consequent is compared with either antecedent or consequent. Thus, if $a:b::c:d,$

then, by division, $a-b:a::c-d:c,$

and $a-b:b::c-d:d.$ See Art. 305.

299. *If four quantities are in proportion, the product of the extremes is equal to the product of the means.*

Let $a:b::c:d.$

Then $\frac{a}{b}=\frac{c}{d},$ Art. 292.

Multiplying each of these equals by bd, we have $ad=bc$.

300. Conversely, *if the product of two quantities is equal to the product of two other quantities, the first two may be made the extremes, and the other two the means of a proportion.*

Let $ad=bc.$

Dividing each of these equals by bd, we have

$$\frac{a}{b}=\frac{c}{d},$$

or $a:b::c:d,$ Art. 292.

EXAMPLES.

1. Given the first three terms of a proportion, 24, 15, and 40, to find the fourth term.

2. Given the first three terms of a proportion, $3ab^3$, $4a^2b^2$, and $9a^3b$, to find the fourth term.

3. Given the last three terms of a proportion, $4a^3b^5$, $3a^3b^3$, and $2a^5b$, to find the first term.

4. Given the first, second, and fourth terms of a proportion, $5y^4$, $7x^2y^3$, and $21x^6y$, to find the third term.

5. Given the first, third, and fourth terms of a proportion, $a+b$, a^2-b^2, and $(a-b)^2$, to find the second term.

Which of the following proportions are correct, and which are incorrect?

6. $3a+4b : 9a+8b :: a-2b : 3a-4b$.

7. $9a^2-4b^2 : 15a^2-25ab+8b^2 :: 15a^2+25ab+8b^2 : 25a^2-16b^2$.

8. $a^3+b^3 : a^2+b^2 :: a^2-b^2 : a-b$.

9. $a^3+b^3 : a+b :: a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5 : a^3-b^3$.

301. *If four quantities are in proportion, they will be in proportion when taken alternately.*

Let $$a : b :: c : d;$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Multiplying by b, $$a=\frac{bc}{d}.$$

Dividing by c, $$\frac{a}{c}=\frac{b}{d},$$

or $$a : c :: b : d.$$

302. *If four quantities are in proportion, they will be in proportion when taken inversely.*

Let $$a : b :: c : d;$$

then $$\frac{a}{b}=\frac{c}{d}.$$

Divide unity by each of these equal quantities, and we have

$$\frac{b}{a}=\frac{d}{c},$$

or $$b : a :: d : c.$$

303. *Ratios that are equal to the same ratio are equal to each other.*

If $a:b::m:n,$ (1.)

and $c:d::m:n,$ (2.)

then $a:b::c:d.$

From proportion (1), $\frac{a}{b}=\frac{m}{n}.$

From proportion (2), $\frac{c}{d}=\frac{m}{n}.$

Hence $\frac{a}{b}=\frac{c}{d},$

or $a:b::c:d.$

304. *If four quantities are proportional, they will be proportional by composition.*

Let $a:b::c:d;$

then $\frac{a}{b}=\frac{c}{d}.$

Add unity to each of these equals, and we have

$$\frac{a}{b}+1=\frac{c}{d}+1;$$

that is, $\frac{a+b}{b}=\frac{c+d}{d},$

or $a+b:b::c+d:d.$

305. *If four quantities are proportional, they will be proportional by division.*

Let $a:b::c:d;$

then $\frac{a}{b}=\frac{c}{d}.$

Subtract unity from each of these equals, and we have

$$\frac{a}{b}-1=\frac{c}{d}-1;$$

that is, $\frac{a-b}{b}=\frac{c-d}{d},$

or $a-b:b::c-d:d.$

306. *If four quantities are proportional, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.*

Let $$a : b :: c : d.$$

By composition, *Art.* 304,

$$a+b : b :: c+d : d.$$

By alternation, *Art.* 301,

$$a+b : c+d :: b : d.$$

Also by division, *Art.* 305,

$$a-b : b :: c-d : d;$$

by alternation, $$a-b : c-d :: b : d.$$

By equality of ratios, *Art.* 303,

$$a+b : c+d :: a-b : c-d,$$

or $$a+b : a-b :: c+d : c-d.$$

307. *If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.*

Let $$a : b :: c : d;$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiply both terms of the first fraction by m, and both terms of the second fraction by n, and we have

$$\frac{ma}{mb} = \frac{nc}{nd},$$

or $$ma : mb :: nc : nd.$$

308. *If four quantities are in proportion, any equimultiples of the antecedents will be proportional to any equimultiples of the consequents.*

Let $$a : b :: c : d;$$

then $$\frac{a}{b} = \frac{c}{d}.$$

Multiply each of these equals by m, and we have

$$\frac{ma}{b} = \frac{mc}{d}.$$

Divide each of these equals by n,

$$\frac{ma}{nb}=\frac{mc}{nd},$$

or $$ma : nb :: mc : nd.$$

309. *If any number of quantities are proportional, any one antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $$a : b :: c : d :: e : f;$$

then, since $$a : b :: c : d,$$

$$ad = bc; \quad (1.)$$

and, since $$a : b :: e : f,$$

$$af = be; \quad (2.)$$

also $$ab = ba. \quad (3.)$$

Adding (1), (2), and (3),

$$ab+ad+af=ba+bc+be;$$

that is, $$a(b+d+f)=b(a+c+e).$$

Hence, *Art.* 300, $a : b :: a+c+e : b+d+f.$

310. *If there are two sets of proportional quantities, the products of the corresponding terms will be proportional.*

Let $$a : b :: c : d,$$

and $$e : f :: g : h;$$

then $$ae : bf :: cg : dh.$$

For $$\frac{a}{b}=\frac{c}{d},$$

and $$\frac{e}{f}=\frac{g}{h}.$$

Multiplying together these equal quantities, we have

$$\frac{ae}{bf}=\frac{cg}{dh},$$

or $$ae : bf :: cg : dh.$$

311. *If four quantities are in proportion, like powers or roots of these quantities will also be in proportion.*

Let $a : b :: c : d;$

then $$\frac{a}{b} = \frac{c}{d}.$$

Raising each of these equals to the nth power, we obtain

$$\frac{a^n}{b^n} = \frac{c^n}{d^n};$$

that is, $a^n : b^n :: c^n : d^n.$

In the same manner, we find

$$a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

312. *If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

If $a : b :: b : c,$

then, by *Art.* 299, $ac = bb = b^2.$

313. *If three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second.*

Let $a : b :: b : c;$

then $$\frac{a}{b} = \frac{b}{c}.$$

Multiply each of these equals by $\frac{a}{b}$, and we have

$$\frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c};$$

that is, $$\frac{a}{c} = \frac{a^2}{b^2},$$

or $a : c :: a^2 : b^2.$

314. *If four quantities are in continued proportion, the first is to the fourth in the triplicate ratio of the first to the second.*

Let $a : b :: b : c :: c : d;$

then $$\frac{a}{b} = \frac{b}{c}, \qquad (1.)$$

and $$\frac{a}{b} = \frac{c}{d}; \qquad (2.)$$

also $$\frac{a}{b} = \frac{a}{b}. \qquad (3.)$$

Multiplying together (1), (2), and (3), we have

$$\frac{a^3}{b^3}=\frac{abc}{bcd}=\frac{a}{d};$$

hence $$a:d::a^3:b^3.$$

VARIATION.

315. Proportions are often expressed in an *abridged form.* Thus, if A and B represent two sums of money put out for one year at the same rate of interest, then

$$\text{A}:\text{B}::\textit{interest of }\text{A}:\textit{interest of }\text{B}.$$

This is briefly expressed by saying that the interest *varies as* the principal. A peculiar character ($\propto$) is used to denote this relation. Thus $\textit{interest} \propto \textit{principal}$

denotes that the interest varies as the principal.

316. One quantity is said to *vary directly* as another when the two quantities increase or decrease together in the same ratio. Thus, in the above example, A varies directly as the interest of A. In such a case, either quantity is equal to the other multiplied by some constant number.

Thus, if the interest *varies as* the principal, then the interest *equals* the product of the principal by some constant number, which is the *rate* of interest.

If $$\text{A} \propto \text{B}, \text{ then } \text{A}=m\text{B}.$$

If the space (S) described by a falling body varies as the square of the time (T), then

$$\text{S}=m\text{T}^2,$$

where m represents a constant multiplier.

317. One quantity may vary directly as *the product of several others.*

Thus, if a body moves with uniform velocity, the space described is measured by the product of the time by the velocity. If we put S to represent the space described, T the time of motion, and V the uniform velocity, then we shall have

$$\text{S} \propto \text{T} \times \text{V}.$$

Also the area of a rectangular figure varies as the product of its length and breadth.

The weight of a stick of timber varies as the product of its length $\times$ its breadth $\times$ its depth $\times$ its density.

318. One quantity is said to *vary inversely* as another when the first varies as the reciprocal of the second. Thus, if the area of a triangle be invariable, the altitude varies inversely as the base.

If the product of two quantities is constant, then one varies inversely as the other.

In uniform motion the space described is measured by the product of the time by the velocity; that is,

$$S \propto T \times V;$$

whence $$T \propto \frac{S}{V}.$$

If the space be supposed to remain constant, then

$$T \propto \frac{1}{V};$$

that is, the time required to travel a given distance varies inversely as the velocity.

Conversely, if one quantity varies inversely as another, the product of the two quantities is *constant.*

Thus, if $$T \propto \frac{1}{V},$$

then the product of T by V is equal to a constant quantity.

319. One quantity is said to vary *directly as a second, and inversely as a third*, when it varies as the product of the second by the reciprocal of the third. Thus, according to the Newtonian law of gravitation, the attraction (G) of any heavenly body varies directly as the quantity of matter (Q), and inversely as the square of the distance (D).

That is, $$G \propto \frac{Q}{D^2}.$$

EXAMPLES.

1. Given $\left\{\begin{array}{c} x+y : x :: 5 : 3 \\ xy=6 \end{array}\right\}$ to find x and y.

Since $$x+y : x :: 5 : 3,$$
by division, *Art.* 305, $$y : x :: 2 : 3.$$

Hence $$3y=2x, \text{ and } y=\frac{2x}{3}.$$

Substituting this value of y in the second equation, we obtain
$$\frac{2x^2}{3}=6.$$

Therefore $$x=\pm 3,$$
and $$y=\pm 2.$$

2. Given $\left\{\begin{array}{c} x+y : x-y :: 3 : 1 \\ x^3-y^3=56 \end{array}\right\}$ to find x and y.

From the first equation, *Art.* 306,
$$2x : 2y :: 4 : 2;$$
whence $$x=2y.$$

Substituting this value of x in the second equation, we find
$$y=2, \text{ and } x=4.$$

3. Given $\left\{\begin{array}{c} (x+y)^2 : (x-y)^2 :: 64 : 1 \\ xy=63 \end{array}\right\}$ to find x and y.

By *Art.* 311, $$x+y : x-y :: 8 : 1.$$
By *Art.* 306, $$2x : 2y :: 9 : 7.$$

Hence $$x=\frac{9y}{7}.$$

Substituting this value of x in the second equation, we find
$$y=\pm 7, \text{ and } x=\pm 9.$$

4. Given $\left\{\begin{array}{c} x^3-y^3 : (x-y)^3 :: 61 : 1 \\ xy=320 \end{array}\right\}$ to find x and y.

From the first equation, by division, *Art.* 305,
$$3xy(x-y) : (x-y)^3 :: 60 : 1.$$

Hence $$960 : (x-y)^2 :: 60 : 1,$$
or $$16 : (x-y)^2 :: 1 : 1.$$

Therefore $x-y=\pm 4.$

Hence $x^2-2xy+y^2=16,$

and $4xy=1280.$

By addition, $x^2+2xy+y^2=1296.$

Hence $x+y=\pm 36.$

Therefore $x=\pm 20$ or $\pm 16,$

and $y=\pm 16$ or $\pm 20.$

5. Given $\left\{\begin{matrix} x^3-y^3 : x^2y-xy^2 :: 7:2 \\ x+y=6 \end{matrix}\right\}$ to find x and y.

Ans. $\left\{\begin{matrix} x=4 \text{ or } 2. \\ y=2 \text{ or } 4. \end{matrix}\right.$

6. Given $\left\{\begin{matrix} \sqrt{y}-\sqrt{a-x}=\sqrt{y-x} \\ \sqrt{y-x}+\sqrt{a-x} : \sqrt{a-x} :: 5:2 \end{matrix}\right\}$ to find x and y.

Ans. $\left\{\begin{matrix} x=\dfrac{4a}{5}. \\ y=\dfrac{5a}{4}. \end{matrix}\right.$

7. Given $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$ to find x.

Ans. $x=9$ or $4.$

8. What number is that to which if 1, 5, and 13 be severally added, the first sum shall be to the second as the second to the third? *Ans.* 3.

9. What number is that to which if a, b, and c be severally added, the first sum shall be to the second as the second to the third? *Ans.* $\dfrac{b^2-ac}{a-2b+c}.$

10. What two numbers are as 2 to 3, to each of which if 4 be added, the sums will be as 5 to 7?

11. What two numbers are as m to n, to each of which if a be added, the sums will be as p to q?

Ans. $\dfrac{am(p-q)}{mq-np}$; $\dfrac{an(p-q)}{mq-np}.$

12. What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5 respectively?

Ans. 2 and 10.

13. What two numbers are those whose difference, sum, and product are as the numbers m, n, and p?

Ans. $\frac{2p}{n+m}$ and $\frac{2p}{n-m}$.

14. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

Ans. 32 and 24.

15. Find two numbers, the greater of which shall be to the less as their sum to a and their difference to b.

Ans. $\frac{(a+b)^2}{2(a-b)}$ and $\frac{a+b}{2}$.

16. There are two numbers which are in the ratio of 3 to 2, the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers. *Ans.* 6 and 4.

17. What two numbers are in the ratio of m to n, the difference of whose fourth powers is to the sum of their cubes as p to q?

Ans. $\frac{mp}{q}\times\frac{m^3+n^3}{m^4-n^4}$, and $\frac{np}{q}\times\frac{m^3+n^3}{m^4-n^4}$.

18. Two circular metallic plates, each an inch thick, whose diameters are 6 and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, admitting that the area of a circle varies as the square of its diameter.

19. Find the radius of a sphere whose volume is equal to the sum of the volumes of three spheres whose radii are 3, 4, and 5 inches, admitting that the volume of a sphere varies as the cube of its radius.

20. Find the radius of a sphere whose volume is equal to the sum of the volumes of three spheres whose radii are r, r', and r''.

CHAPTER XVI.

PROGRESSIONS.

ARITHMETICAL PROGRESSION.

320. *An arithmetical progression* is a series of quantities which increase or decrease by a common difference. Thus the following series are in arithmetical progression:

$$1, 3, 5, 7, 9, \ldots$$
$$20, 17, 14, 11, 8, \ldots$$
$$a, a+d, a+2d, a+3d, \ldots$$
$$a, a-d, a-2d, a-3d, \ldots$$

In the first example the common difference is $+2$, and the series forms an *increasing* arithmetical progression; in the second example the common difference is -3, and the series forms a *decreasing* arithmetical progression. In the third example the common difference is $+d$, and in the fourth example it is $-d$.

321. In an arithmetical progression having a finite number of terms, there are *five quantities* to be considered, viz., the first term, the last term, the number of terms, the common difference, and the sum of the terms. When any three of them are given, the other two may be found. We will denote

the first term by	a,
the last term by	l,
the number of terms by	n,
the common difference by	d,
and the sum of the terms by	s.

The first term and the last term are called the *extremes*, and all the other terms are called *arithmetical means*.

322. *In an arithmetical progression the last term is equal to the first term plus the product of the common difference by the number of terms less one.*

Let the terms of the series be represented by

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \text{ etc.}$$

Since the coefficient of d in the *second* term is 1, in the *third* term 2, in the *fourth* term 3, and so on, the nth term of the series will be

$$a+(n-1)d,$$

or

$$l=a+(n-1)d,$$

in which d is positive or negative according as the series is an increasing or a decreasing one.

323. *The sum of any number of terms in arithmetical progression is equal to one half the sum of the two extremes multiplied by the number of terms.*

The term preceding the last will be $l-d$, the term preceding that $l-2d$, and so on. If the terms of the series be written in the reverse order, the sum will be the same as when written in the direct order. Hence we have

$$s=a+(a+d)+(a+2d)+(a+3d)+\ldots\ldots +l,$$
$$s=l+(l-d)+(l-2d)+(l-3d)+\ldots\ldots +a.$$

Adding these equations term by term, we have

$$2s=(a+l)+(a+l)+(a+l)+\ldots\ldots +(a+l).$$

Here $a+l$ is taken n times; hence

$$2s=n(a+l),$$

or

$$s=\frac{n}{2}(a+l).$$

324. *In an arithmetical progression the sum of the extremes is equal to the sum of any two terms equidistant from the extremes.*

This principle follows from the preceding demonstration. It may also be shown independently as follows:

The mth term from the beginning is $a+(m-1)d$.

The mth term from the end is $l-(m-1)d$.

And the sum of these terms is $a+l$.

325. *To insert any number of arithmetical means between two given terms.*

The whole number of terms in the series consists of the two extremes and all the intermediate terms. If, then, m represents the number of means, $m+2$ will be the whole number of terms.

Substituting $m+2$ for n in the formula, Art. 322, we have

$$l=a+(m+1)d,$$

or $$d=\frac{l-a}{m+1}=\text{the common difference},$$

whence the required means are easily obtained by addition.

326. The two equations

$$l=a+(n-1)d,$$
$$s=\frac{n}{2}(a+l),$$

contain five quantities, a, l, n, d, s, of which any three being given, the other two can be found. We may therefore have *ten different cases*, each requiring the determination of two different formulæ. These formulæ are exhibited in the following table, and should be verified by the student.

No.	Given.	Required.	Formulæ.	
1.	$a, d, n,$	$l, s,$	$l=a+(n-1)d;$	$s=\frac{1}{2}n[2a+(n-1)d].$
2.	$l, d, n,$	$a, s,$	$a=l-(n-1)d;$	$s=\frac{1}{2}n[2l-(n-1)d].$
3.	$a, l, n,$	$d, s,$	$d=\frac{l-a}{n-1};$	$s=\frac{1}{2}n(a+l).$
4.	$a, n, s,$	$d, l,$	$d=\frac{2s-2an}{n(n-1)};$	$l=\frac{2s}{n}-a.$
5.	$n, d, s,$	$a, l,$	$a=\frac{s}{n}-\frac{(n-1)d}{2};$	$l=\frac{s}{n}+\frac{(n-1)d}{2}.$
6.	$l, n, s,$	$a, d,$	$a=\frac{2s}{n}-l;$	$d=\frac{2nl-2s}{n(n-1)}.$
7.	$a, d, l,$	$n, s,$	$n=\frac{l-a}{d}+1;$	$s=\frac{(l+a)(l-a+d)}{2d}.$
8.	$a, l, s,$	$n, d,$	$n=\frac{2s}{a+l};$	$d=\frac{l^2-a^2}{2s-a-l}.$
9.	$a, d, s,$	$l, n,$	$l=-\frac{1}{2}d\pm\sqrt{2ds+(a-\frac{1}{2}d)^2};$	$n=\frac{d-2a\pm\sqrt{(2a-d)^2+8ds}}{2d}.$
10.	$l, d, s,$	$a, n,$	$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2ds};$	$n=\frac{2l+d\pm\sqrt{(2l+d)^2-8ds}}{2d}.$

EXAMPLES.

1. The first term of an arithmetical progression is 2, and the common difference is 4; what is the 10th term?

Ans. 38.

2. The first term is 40, and the common difference -3; what is the 10th term?

3. The first term is 1, and the common difference $\frac{3}{4}$; what is the 10th term?

4. The first term is 1, and the common difference $-\frac{1}{6}$; what is the 10th term?

5. The first term is 5, the common difference is 10, and the number of terms is 60; what is their sum?

Ans. 18000.

6. The first term is 116, the common difference is -4, and the number of terms is 25; what is their sum?

7. The first term is 1, the common difference is $\frac{3}{4}$, and the number of terms is 12; what is their sum?

8. The first term is $1\frac{3}{5}$, the common difference is $-\frac{2}{5}$; and the number of terms is 10; what is their sum?

9. Required the number of terms of a progression whose sum is 442, whose first term is 2, and common difference 3.

Ans. 17.

10. Required the first term of a progression whose sum is 99, whose last term is 19, and common difference 2.

11. The sum of a progression is 1455, the first term 5, and the last term 92; what is the common difference?

12. Required the sum of 101 times of the series

1, 3, 5, 7, 9, etc. *Ans.* 10201.

13. Find the nth term of the series

1, 3, 5, 7, 9, etc. *Ans.* $2n-1$.

14. Find the sum of n terms of the series

1, 3, 5, 7, 9, etc. *Ans.* n^2.

15. Find the sum of n terms of the series of numbers

1, 2, 3, 4, 5, etc.

Ans. $\frac{n(n+1)}{2}$.

16. Find the sum of n terms of the series

2, 4, 6, 8, etc. *Ans.* $n(n+1)$.

17. Find 6 arithmetical means between 1 and 50.

18. Find 7 arithmetical means between $\frac{1}{3}$ and 3.

19. A body falls 16 feet during the first second, and in each succeeding second 32 feet more than in the one immediately preceding; if it continue falling for 20 seconds, how many feet will it pass over in the last second, and how many in the whole time?

Ans. 624 feet in the last second, and 6400 feet in the whole time.

20. One hundred stones being placed on the ground in a straight line at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket which is placed two yards from the first stone?

Ans. 20200 yards.

PROBLEMS.

327. When of the five quantities a, l, n, d, s, no three are directly given, it may be necessary to represent the series by the use of two unknown quantities. The form of the series which will be found most convenient will depend upon the conditions of the problem. If x denote the first term and y the common difference, then

$$x,\ x+y,\ x+2y,\ x+3y, \text{ etc.},$$

will represent a series in arithmetical progression.

It will, however, generally be found most convenient to represent the series in such a manner that the common difference may disappear in taking the sum of the terms. Thus a progression of three terms may be represented by

$$x-y,\ x,\ x+y;$$

one of four terms by $x-3y$, $x-y$, $x+y$, $x+3y$;

one of five terms by $x-2y$, $x-y$, x, $x+y$, $x+2y$.

Prob. 1. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 26; and if 198 be added to it, the digits will be inverted; required the number. *Ans.* 234.

Prob. 2. Find three numbers in arithmetical progression the sum of whose squares shall be 1232, and the square of the mean greater than the product of the two extremes by 16.

Ans. 16, 20, and 24.

Prob. 3. Find three numbers in arithmetical progression the sum of whose squares shall be a, and the square of the mean greater than the product of the two extremes by b.

$$Ans.\ \sqrt{\frac{a-2b}{3}}-\sqrt{b};\ \sqrt{\frac{a-2b}{3}};\ \text{and}\ \sqrt{\frac{a-2b}{3}}+\sqrt{b}.$$

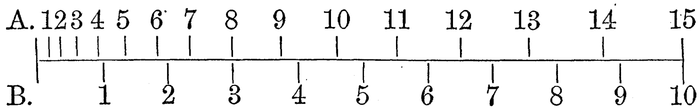
Prob. 4. Find four numbers in arithmetical progression whose sum is 28, and continued product 585.

Ans. 1, 5, 9, 13.

Prob. 5. A sets out for a certain place, and travels 1 mile the first day, 2 the second, 3 the third, and so on. In five days afterward B sets out, and travels 12 miles a day. How long will A travel before he is overtaken by B?

Ans. 8 or 15 days.

This is another example of an equation of the second degree, in which the two roots are both positive. The following diagram exhibits the daily progress of each traveler. The divisions above the horizontal line represent the distances traveled each day by A; those below the line the distances traveled by B.



It is readily seen from the figure that A is in advance of B until the end of his 8th day, when B overtakes and passes him. After the 12th day, A gains upon B, and passes him on the 15th day, after which he is continually gaining upon B, and could not be again overtaken.

Prob. 6. A goes 1 mile the first day, 2 the second, and so on. B starts a days later, and travels b miles per day. How long will A travel before he is overtaken by B?

$$Ans.\ \frac{2b-1\pm\sqrt{(2b-1)^2-8ab}}{2}\ \text{days}.$$

In what case would B *never* overtake A?

Ans. When $a > \frac{(2b-1)^2}{8b}$.

For instance, in the preceding example, if B had started one day later, he could never have overtaken A.

Prob. 7. A traveler set out from a certain place and went 1 mile the first day, 3 the second, 5 the third, and so on. After he had been gone three days, a second traveler sets out, and goes 12 miles the first day, 13 the second, and so on. After how many days will they be together?

Ans. In 2 or 9 days.

Let the student illustrate this example by a diagram like the preceding.

Prob. 8. A and B, 165 miles distant from each other, set out with a design to meet. A travels 1 mile the first day, 2 the second, 3 the third, and so on. B travels 20 miles the first day, 18 the second, 16 the third, and so on. In how many days will they meet? *Ans.* 10 or 33 days.

GEOMETRICAL PROGRESSION.

328. *A geometrical progression is a series of quantities each of which is equal to the product of the preceding one by a constant factor.*

The constant factor is called the *ratio* of the series.

329. When the first term is positive, and the ratio greater than unity, the series forms an *increasing* geometrical progression, as

$$2, 4, 8, 16, 32, \text{etc.},$$

in which the ratio is 2.

When the ratio is less than unity, the series forms a *decreasing* geometrical progression, as

$$81, 27, 9, 3, \text{etc.},$$

in which the ratio is $\frac{1}{3}$.

330. In a geometrical progression having a finite number of terms, there are *five quantities* to be considered, viz., the first term, the last term, the number of terms, the ratio, and the sum of the terms. When any three of these are given, the other two may be found. We will denote

the first term by a,
the last term by l,
the number of terms by n,
the ratio by r,
and the sum of the terms by s.

The first term and the last term are called the *extremes*, and all the other terms are called *geometrical means*.

331. *In a geometrical progression, the last term is equal to the product of the first term by that power of the ratio whose exponent is one less than the number of terms.*

According to the definition, the second term is equal to the first multiplied by r, that is, it is equal to ar; the third term is equal to the second multiplied by r, that is, it is equal to ar^2; the fourth term is equal to the third multiplied by r, that is, it is equal to ar^3; and so on. Hence the nth term of the series will be equal to ar^{n-1}; hence we shall have

$$l=ar^{n-1}.$$

332. To find the sum of any number of terms in geometrical progression, *multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less one.*

From the definition, we have

$$s=a+ar+ar^2+\ldots\ldots+ar^{n-2}+ar^{n-1}.$$

Multiplying this equation by r, we have

$$rs=ar+ar^2+\ldots\ldots+ar^{n-2}+ar^{n-1}+ar^n.$$

Subtracting the first equation from the second, member from member, we have $rs-s=ar^n-a.$

Hence $$s=\frac{ar^n-a}{r-1};$$

or, substituting the value of l already found, we have

$$s=\frac{rl-a}{r-1}.$$

If we had subtracted the second equation from the first, we should have found

$$s=\frac{a-rl}{1-r},$$

which is the most convenient formula when r is less than unity, and the series is, therefore, a decreasing one.

333. To find the sum of a decreasing geometrical series when the number of terms is infinite, *divide the first term by unity diminished by the ratio.*

The sum of the terms of a decreasing series may be represented by the formula

$$s=\frac{a-rl}{1-r}.$$

Now, in a decreasing series, each term is less than the preceding, and the greater the number of terms, the smaller will be the last term of the series. If the number of terms be infinite, the last term of the series will be less than any assignable number, and rl may be neglected in comparison with a. In this case the formula reduces to

$$s=\frac{a}{1-r}.$$

334. *To find any number of geometrical means between two given terms.*

In order to solve this problem, it is necessary to know the *ratio.* If m represent the number of means, $m+2$ will be the whole number of *terms.* Hence, putting $m+2$ for n in the formula, Art. 331, we have

$$l=ar^{m+1};$$

whence we obtain $$r=\left(\frac{l}{a}\right)^{\frac{1}{m+1}}.$$

That is, to find the ratio, *divide the last term by the first term, and extract the root which is denoted by the number of means plus one.* Having found the ratio, the required means may be obtained by continued multiplication.

335. The two equations

$$l=ar^{n-1}, \quad s=\frac{ar^n-a}{r-1},$$

contain five quantities, a, l, n, r, s, of which any three being given, the other two can be found. We may therefore have *ten different cases*, each requiring the determination of two quantities, thus giving rise to twenty different formulæ. The first four of the following cases are readily solved. The fifth and sixth cases involve the solution of equations of a higher degree than the second. When n is not large, the value of the unknown quantity can generally be found by a few trials. The four remaining cases, when n is the quantity sought, involve the solution of an exponential equation. See Art. 416. These different cases are all exhibited in the following table for convenient reference.

No.	Given.	Required.	Formulæ.	
1.	$a, r, n,$	$l, s,$	$l=ar^{n-1}$;	$s=\frac{ar^n-a}{r-1}$.
2.	$l, r, n,$	$a, s,$	$a=\frac{l}{r^{n-1}}$;	$s=\frac{lr^n-l}{r^n-r^{n-1}}$.
3.	$n, r, s,$	$a, l,$	$a=\frac{(r-1)s}{r^n-1}$;	$l=\frac{(r-1)sr^{n-1}}{r^n-1}$.
4.	$a, l, n,$	$r, s,$	$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$;	$s=\frac{l^{\frac{n}{n-1}}-a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}}-a^{\frac{1}{n-1}}}$.
5.	$a, n, s,$	$r, l,$	$ar^n-rs=a-s$;	$l(s-l)^{n-1}=a(s-a)^{n-1}$.
6.	$l, n, s,$	$a, r,$	$a(s-a)^{n-1}=l(s-l)^{n-1}$;	$(s-l)r^n-sr^{n-1}=-l$.
7.	$a, r, l,$	$s, n,$	$s=\frac{lr-a}{r-1}$;	$n=\frac{\log. l-\log. a}{\log. r}+1$.
8.	$a, l, s,$	$r, n,$	$r=\frac{s-a}{s-l}$;	$n=\frac{\log. l-\log. a}{\log. (s-a)-\log. (s-l)}+1$.
9.	$a, r, s,$	$l, n,$	$l=\frac{a+(r-1)s}{r}$;	$n=\frac{\log. [a+(r-1)s]-\log. a}{\log. r}$.
10.	$l, r, s,$	$a, n,$	$a=lr-(r-1)s$;	$n=\frac{\log. l-\log. [lr-(r-1)s]}{\log. r}+1$.

EXAMPLES.

1. Find the 12th term of the series 1, 3, 9, 27, etc.

We have $l = ar^{n-1} = 3^{11} = 177147$, *Ans.*

2. Given the first term 2, the ratio 3, and the number of terms 10; to find the last term. *Ans.* 39366.

3. Find the sum of 14 terms of the series 1, 2, 4, 8, 16, etc.

$$s = \frac{ar^n - a}{r-1} = 2^{14} - 1 = 16383, \textit{ Ans.}$$

4. Find the sum of 12 terms of the series 1, 3, 9, 27, etc. *Ans.* 265,720.

5. Given the first term 1, the last term 512, and the sum of the terms 1023; to find the ratio.

6. Given the last term 2048, the number of terms 12, and the ratio 2; to find the first term.

7. Find the sum of 6 terms of the series 6, $4\frac{1}{2}$, $3\frac{3}{8}$, etc. *Ans.* $19\frac{373}{512}$.

8. Find the sum of 15 terms of the series 8, 4, 2, 1, etc. *Ans.* $15\frac{2047}{2048}$.

9. Find three geometrical means between 2 and 162.

10. Find two geometrical means between 4 and 256.

11. Find three geometrical means between a and b. *Ans.* $\sqrt[4]{a^3b}$, $\sqrt{ab}$, $\sqrt[4]{ab^3}$.

12. Find the value of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$, etc., to infinity.

$$s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2, \textit{ Ans.}$$

13. Find the value of $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$, etc., to infinity. *Ans.* $\frac{3}{2}$.

14. Find the value of $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+$, etc., to infinity.

15. Find the ratio of an infinite progression whose first term is 1, and the sum of the series $\frac{5}{4}$. *Ans.* $\frac{1}{5}$.

16. Find the first term of an infinite progression whose ratio is $\frac{1}{10}$, and the sum $\frac{2}{3}$. *Ans.* $\frac{3}{5}$.

17. Find the first term of an infinite progression of which the ratio is $\frac{1}{n}$, and the sum $\frac{n}{n-1}$.

18. Find the value of the series $3+2+\frac{4}{3}+$, etc., to infinity.

19. Find the value of the series $\frac{4}{3}+1+\frac{3}{4}+$, etc., to infinity.

20. A gentleman, being asked to dispose of his horse, said he would sell him on condition of receiving one cent for the first nail in his shoes, two cents for the second, and so on, doubling the price of every nail to 32, the number of nails in his four shoes. What would the horse cost at that rate?

Ans. \$42,949,672.95.

PROBLEMS.

Prob. 1. Find three numbers in geometrical progression such that their sum shall be 21, and the sum of their squares 189.

Denote the first term by x and the ratio by y; then

$$x + xy + xy^2 = 21, \qquad (1.)$$

$$x^2 + x^2y^2 + x^2y^4 = 189. \qquad (2.)$$

Transposing xy in Eq. (2), squaring, and reducing, we have

$$x^2 + x^2y^2 + x^2y^4 = 441 - 42xy. \qquad (3.)$$

Comparing (2) and (3), $xy = 6$, or $x = \frac{6}{y}$.

Substituting this value of x in Eq. (1), and reducing, we have

$$y^2 - \frac{5y}{2} = -1.$$

Whence $y = 2$ or $\frac{1}{2}$, and $x = 3$ or 12.

The terms are therefore 3, 6, and 12, or 12, 6, and 3.

Prob. 2. Find four numbers in geometrical progression such that the sum of the first and second shall be 15, and the sum of the third and fourth 60.

By the conditions, $x + xy = 15, \qquad (1.)$

$$xy^2 + xy^3 = 60. \qquad (2.)$$

Multiplying Eq. (1) by y^2, we have

$$xy^2 + xy^3 = 15y^2 = 60.$$

Therefore $y^2 = 4$, and $y = \pm 2$.

Also $x \pm 2x = 15$;

therefore $x = 5$ or -15.

Taking the first value of x and the corresponding value of y, we obtain the series 5, 10, 20, and 40.

Taking the second value of x and the corresponding value of y, we obtain the series -15, $+30$, -60, and $+120$;

which numbers also perfectly satisfy the problem understood algebraically. If, however, it is required that the terms of the progression be positive, the last value of x would be inapplicable to the problem, though satisfying the algebraic equation.

Several of the following problems also have two solutions, if we admit negative values.

Prob. 3. Find three numbers in geometrical progression such that their sum shall be 210, and the last shall exceed the first by 90. *Ans.* 30, 60, and 120.

Prob. 4. Find three numbers in geometrical progression such that their sum shall be 42, and the sum of the first and last shall be 34. *Ans.* 2, 8, and 32.

Prob. 5. Find three numbers in geometrical progression such that their continued product may be 64, and the sum of their cubes 584. *Ans.* 2, 4, and 8.

Prob. 6. Find four numbers in geometrical progression such that the difference between the first and second may be 4, and the difference between the third and fourth 36.

Ans. 2, 6, 18, and 54.

Prob. 7. Find four numbers in geometrical progression such that the sum of the first and third may be a, and the sum of the second and fourth may be b.

Ans. $\frac{a^3}{a^2+b^2}$, $\frac{a^2b}{a^2+b^2}$, $\frac{ab^2}{a^2+b^2}$, and $\frac{b^3}{a^2+b^2}$.

Prob. 8. Find four numbers in geometrical progression such that the fourth shall exceed the second by 24, and the sum of the extremes shall be to the sum of the means as 7 to 3.

Ans. 1, 3, 9, and 27.

Prob. 9. The sum of $700 was divided among four persons, whose shares were in geometrical progression, and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares? *Ans.* 108, 144, 192, and 256.

Prob. 10. Find six numbers in geometrical progression such that their sum shall be 1365, and the sum of the third and fourth shall be 80. *Ans.* 1, 4, 16, 64, 256, and 1024.

CHAPTER XVII.

CONTINUED FRACTIONS.—PERMUTATIONS AND COMBINATIONS.

336. *A continued fraction* is one whose numerator is unity, and its denominator an integer plus a fraction, whose numerator is likewise unity, and its denominator an integer plus a fraction, and so on.

The general form of a continued fraction is

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{, etc.}}}}}$$

When the number of terms a, b, c, etc., is *finite*, the continued fraction is said to be *terminating;* such a continued fraction may be reduced to an ordinary fraction by performing the operations indicated.

337. *To convert any given fraction into a continued fraction.*

Let $\frac{m}{n}$ be the given fraction; divide m by n; let A be the quotient, and p the remainder: thus,

$$\frac{m}{n}=\text{A}+\frac{p}{n}=\text{A}+\frac{1}{\frac{n}{p}}.$$

Divide n by p; let a be the quotient, and q the remainder: thus,

$$\frac{n}{p}=a+\frac{q}{p}=a+\frac{1}{\frac{p}{q}}.$$

Similarly, $$\frac{p}{q}=b+\frac{r}{q}=b+\frac{1}{\frac{q}{r}};$$

and so on, so that we have

$$\frac{m}{n}=\mathrm{A}+\cfrac{1}{a+\cfrac{1}{b+,\text{ etc.}}}$$

We see, then, that to convert a given fraction into a continued fraction, we proceed as if we were finding the greatest common divisor of the numerator and denominator; and we must, therefore, at last arrive at a point where the remainder is zero, and the operation terminates; hence every rational fraction can be converted into a *terminating* continued fraction.

Ex. 1. Transform $\frac{114}{347}$ into a continued fraction.

$$\textit{Ans. } \cfrac{1}{3+\cfrac{1}{22+\cfrac{1}{1+\frac{1}{4}}}}.$$

Ex. 2. Transform $\frac{246}{372}$ into a continued fraction.

$$\textit{Ans. } \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{20}}}}.$$

Ex. 3. Transform $\frac{351}{965}$ into a continued fraction.

$$\textit{Ans. } \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{87}}}}}.$$

Ex. 4. Transform $\frac{421}{972}$ into a continued fraction.

Ex. 5. Transform $\frac{251}{764}$ into a continued fraction.

Ex. 6. Transform $\frac{130}{421}$ into a continued fraction.

338. *To find the value of a terminating continued fraction.*

Ex. 1. Find the value of the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}.$$

Beginning with the last fraction, we have

$$3+\tfrac{1}{4}=\tfrac{13}{4}.$$

Hence $$\frac{1}{3+\frac{1}{4}}=\tfrac{4}{13}.$$

Therefore $$2+\frac{1}{3+\frac{1}{4}}=\tfrac{30}{13},$$

and $$\cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}}=\tfrac{13}{30},\ Ans.$$

Ex. 2. Find the value of the continued fraction

$$\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{5}}}}.$$

Ex. 3. Find the value of the continued fraction

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}}}.$$

Ex. 4. Find the value of the continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{55}}}}}}.$$

339. *To find the value of an infinite continued fraction.*

Let the fraction be

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+, \text{ etc.}}}}$$

An approximate value of this fraction is obtained by omitting all its terms beyond any assumed fraction, and obtaining the value of the resulting fraction, as in the previous article.

Thus we obtain

1st approximate value, $\frac{1}{a}$;

2d approximate value, $\dfrac{1}{a+\dfrac{1}{b}}=\dfrac{b}{ab+1}$;

3d approximate value, $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}}=\dfrac{bc+1}{(ab+1)c+a}$;

4th approximate value, $\dfrac{(bc+1)d+b}{(ab+1)cd+ad+ab+1}$, etc.

340. The fractions formed by taking one, two, three, etc., of the quotients of the continued fraction are called *converging fractions*, or *convergents*.

The convergents, taken in order, are *alternately less and greater* than the continued fraction.

The first convergent $\frac{1}{a}$ is *too great*, because the denominator a is too small; the second convergent $\frac{b}{ab+1}$ is *too small*, because $a+\frac{1}{b}$ is too great, and so on.

341. When a fraction has been transformed into a continued fraction, its *approximate value* may be found by taking a few of the first terms of the continued fraction.

Thus an approximate value of $\frac{114}{347}$ is $\frac{1}{3}$, which is the first term of its continued fraction.

By taking two terms, we obtain $\frac{22}{67}$, which is a nearer approximation; and three terms would give a still more accurate value.

Ex. 1. Find approximate values of the fraction $\frac{532}{1193}$.

Ans. $\frac{1}{2}$, $\frac{4}{9}$, $\frac{33}{74}$.

Ex. 2. Find approximate values of the fraction $\frac{115}{424}$.

Ex. 3. Find approximate values of the fraction $\frac{119}{409}$.

342. By the preceding method we are enabled to discover the approximate value of a fraction expressed in large numbers, and this principle has some important applications, particularly in Astronomy.

Ex. 4. The ratio of the circumference of a circle to its diameter is 3.1415926. Find approximate values for this ratio.

Ans. $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$.

Ex. 5. The length of the tropical year is 365*d.* 5*h.* 48*m.* 48*s.* Find approximate values for the ratio of 5*h.* 48*m.* 48*s.* to 24 hours.

Ans. $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$.

Ex. 6. In 87969 years the earth makes 277287 conjunctions with Mercury. Find approximate values for the ratio of 87969 to 277287.

Ans. $\frac{1}{3}$, $\frac{6}{19}$, $\frac{7}{22}$, $\frac{13}{41}$, $\frac{33}{104}$.

Ex. 7. In 57551 years the earth makes 36000 conjunctions with Venus. Find approximate values for the ratio of 57551 to 36000.

Ans. $\frac{8}{5}$, $\frac{235}{147}$.

Ex. 8. In 295306 years the moon makes 3652422 synodical revolutions. Find an approximate value for the ratio of 295306 to 3652422. *Ans.* $\frac{19}{235}$.

Ex. 9. One French metre is equal to 3.2809 English feet. Find approximate values for the ratio of a metre to a foot.

Ex. 10. One French kilogramme is equal to 2.2046 pounds avoirdupois. Find approximate values for the ratio of a kilogramme to a pound.

Ex. 11. One French litre is equal to 0.2201 English gallons. Find approximate values for the ratio of a litre to a gallon.

PERMUTATIONS AND COMBINATIONS.

343. The different orders in which things can be arranged are called their *permutations.* In forming permutations, all of the things or a part only may be taken at a time.

Thus the permutations of the three letters *a*, *b*, *c*, taken *all together*, are

abc, *acb*, *bac*, *bca*, *cab*, *cba*.

The permutations of the same letters taken *two at a time* are

$$ab, ac, ba, bc, ca, cb.$$

The permutations of the same letters taken *one at a time* are

$$a, b, c.$$

344. *The number of permutations of n things taken m at a time is equal to the continued product of the natural series of numbers from n down to $n-m+1$.*

Suppose the things to be n letters, $a, b, c, d \ldots\ldots$

The number of permutations of n letters, taken singly or one at a time, is evidently equal to the number of letters, or to n.

If we wish to form all the permutations of n letters taken two at a time, we must write after each letter each of the $n-1$ remaining letters. We shall thus obtain $n(n-1)$ permutations.

If we wish to form all the permutations of n letters taken three at a time, we must write after each of the permutations of n letters taken two at a time each of the $n-2$ remaining letters. We shall thus obtain $n(n-1)(n-2)$ permutations.

In the same manner we shall find that the number of permutations of n letters taken four at a time is

$$n(n-1)(n-2)(n-3).$$

Hence we may conclude that the number of permutations of n letters taken m at a time is

$$n(n-1)(n-2)(n-3)\ldots\ldots(n-m+1).$$

345. *The number of permutations of n things taken all together is equal to the continued product of the natural series of numbers from 1 to n.*

If we suppose that each permutation comprehends all the n letters; that is, if $m=n$, the preceding formula becomes

$$n(n-1)(n-2)\ldots\ldots 3\times 2\times 1;$$

or, inverting the order of the factors,

$$1.2.3.4\ldots\ldots(n-1)n,$$

which expresses the number of permutations of n things taken all together.

For the sake of brevity, $1.2.3.4 \ldots\ldots (n-1)n$ is often denoted by $\underline{|n}$; that is, $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive.

346. The *combinations* of things are the different collections which can be formed out of them without regarding the *order* in which the things are placed.

Thus the three letters a, b, c, taken all together, form but one combination, abc.

Taken two and two, they form three combinations, ab, ac, bc.

347. *The number of combinations of n things, taken m at a time, is equal to the continued product of the natural series of numbers from n down to* $n-m+1$ *divided by the continued product of the natural series of numbers from* 1 *to m.*

The number of combinations of n letters taken separately, or one at a time, is evidently n.

The number of combinations of n letters taken two at a time is $\frac{n(n-1)}{1.2}$.

For the number of permutations of n letters taken two at a time is $n(n-1)$, and there are two permutations (ab, ba) corresponding to one combination of two letters; therefore the number of combinations will be found by dividing the number of permutations by 2.

The number of combinations of n letters taken three at a time is $\frac{n(n-1)(n-2)}{1.2.3}$.

For the number of permutations of n letters taken three at a time is $n(n-1)(n-2)$, and there are $1.2.3$ permutations for one combination of these letters; therefore the number of combinations will be found by dividing the number of permutations by $1.2.3$.

In the same manner we shall find the number of combinations of n letters taken m at a time to be

$$\frac{n(n-1)(n-2) \ldots\ldots (n-m+1)}{1.2.3 \ldots\ldots m}.$$

EXAMPLES.

1. How many different permutations may be formed of 8 letters taken 5 at a time? *Ans.* $8.7.6.5.4=6720$.

2. How many different permutations may be formed of the 26 letters of the alphabet taken 4 at a time?

Ans. 358800.

3. How many different permutations may be formed of 12 letters taken 6 at a time? *Ans.* 665280.

4. How many different permutations may be formed of 8 things taken all together?

Ans. $1.2.3.4.5.6.7.8=40320$.

5. How many different permutations may be made of the letters in the word *Roma* taken all together?

6. How many different permutations may be made of the letters in the word *virtue* taken all together?

7. What is the number of different arrangements which can be formed of 12 persons at a dinner-table?

Ans. 479001600.

8. How many different combinations may be formed of 6 letters taken 3 at a time?

Ans. $\frac{6.5.4}{1.2.3}=20$.

9. How many different combinations may be formed of 8 letters taken 4 at a time? *Ans.* 70.

10. How many different combinations may be formed of 10 letters taken 6 at a time? *Ans.* 210.

11. A telegraph has m arms, and each arm is capable of n distinct positions; find the total number of signals which can be made with the telegraph.

12. How many different numbers can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, each of these digits occurring once, and only once, in each number?

CHAPTER XVIII.

BINOMIAL THEOREM.

348. The *binomial theorem*, or *binomial formula*, is a formula discovered by Newton, by means of which we may obtain any power of a binomial $x+a$, without obtaining the preceding powers.

349. By actual multiplication, we find the successive powers of $x+a$ to be as follows:

$$(x+a)^2=x^2+2ax+a^2,$$
$$(x+a)^3=x^3+3ax^2+3a^2x+a^3,$$
$$(x+a)^4=x^4+4ax^3+6a^2x^2+4a^3x+a^4,$$
$$(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5.$$

The powers of $x-a$, found in the same manner, are as follows:

$$(x-a)^2=x^2-2ax+a^2,$$
$$(x-a)^3=x^3-3ax^2+3a^2x-a^3,$$
$$(x-a)^4=x^4-4ax^3+6a^2x^2-4a^3x+a^4,$$
$$(x-a)^5=x^5-5ax^4+10a^2x^3-10a^3x^2+5a^4x-a^5.$$

On comparing the powers of $x+a$ with those of $x-a$, we perceive that they only differ in *the signs of certain terms.* In the powers of $x+a$, all the terms are positive. In the powers of $x-a$, the terms containing the odd powers of a have the sign minus, while the terms containing the even powers have the sign plus. The reason of this is obvious; for, since $-a$ is the only negative term of the root, the terms of the power can only be rendered negative by a. A term which contains the factor $-a$ an even number of times will therefore be positive; if it contain it an odd number of times it must be negative. Hence it appears that it is only necessary to seek for a method of obtaining the powers of $x+a$, for these will become the powers of $x-a$ by simply changing the signs of the alternate terms.

350. *Law of the Exponents.*—The exponents of x and of a in the different powers follow a simple law. In the first term of each power, x is raised to the required power of the binomial; and in the following terms the exponents of x continually decrease by unity to zero, while the exponents of a increase by unity from zero up to the required power of the binomial.

351. *Law of the Coefficients.*—The coefficient of the first term is unity; that of the second term is the exponent of the power; and the coefficients of terms equidistant from the extremes are equal to each other; but after the first two terms it is not obvious how to obtain the coefficients of the fourth and higher powers.

In order to discover the law of the coefficients, we will form the product of several binomial factors whose second terms are all different; thus,

$$(x+a)(x+b)=\begin{array}{r|l} x^2+a & x+ab. \\ +b & \end{array}$$

$$(x+a)(x+b)(x+c)=\begin{array}{r|r|l} x^3+a & x^2+ab & x+abc. \\ +b & +ac & \\ +c & +bc & \end{array}$$

$$(x+a)(x+b)(x+c)(x+d)=\begin{array}{r|r|r|l} x^4+a & x^3+ab & x^2+abc & x+abcd. \\ +b & +ac & +abd & \\ +c & +bc & +acd & \\ +d & +ad & +bcd & \\ & +bd & & \\ & +cd & & \end{array}$$

In each of these products the exponent of x in the first term is equal to the number of binomial factors, and in the following terms continually decreases by one. *The coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of all their products taken two and two*, and so on. *The last term is the product of the second terms of the binomial factors.*

352. We will now prove that *if the laws of formation just stated are true for any power, they will also hold true for the formation of the next higher power.*

Suppose that we have found the product of m binomials $x+a$, $x+b$, ... $x+k$. Let P_1 denote the sum of the second terms of the binomials, P_2 the sum of the different products of these second terms taken two and two, P_3 the sum of their products taken three and three, and so on; and let P_m denote the product of all these second terms. The product of the given binomials will then be

$$x^m+P_1x^{m-1}+P_2x^{m-2}+P_3x^{m-3}\ \ldots\ldots\ +P_m.$$

Multiplying this polynomial by a new binomial, $x+l$, we obtain the following product:

$$\begin{array}{r|r|r|r|l} x^{m+1}+P_1 & x^m+P_2 & x^{m-1}+P_3 & x^{m-2}\ldots\ \ +P_m & x \\ +l & +lP_1 & +lP_2 & +lP_{m-1} & +lP_m. \end{array}$$

The law of the exponents of x remains the same. The coefficient of the first term is still equal to unity, and that of the second term is the sum of the second terms of the $m+1$ binomials. The coefficient of the third term consists of the sum of the products of the second terms of the m binomial factors taken two and two, increased by the sum of the same second terms multiplied by l, which is equivalent to the sum of the products of the second terms of the $m+1$ binomials taken two and two. The coefficient of the fourth term consists of the sum of the products of the second terms of the m factors of the first product taken three and three, increased by the sum of the products of their second terms taken two and two multiplied by l, which is equivalent to the sum of the products of the second terms of the $m+1$ binomials taken three and three, and so on. The last term is equal to the product of the second terms of the m binomial factors multiplied by l, which is equivalent to the product of the second terms of the $m+1$ binomials.

Hence the law which was supposed true for m factors is true for $m+1$ factors; and therefore, since it has been verified for two factors, it is true for three; being true for three factors, it is also true for four, and so on; therefore the law is general.

353. *Powers of a Binomial.*—If now, in the preceding binomial factors, we suppose the second terms to be all equal to a, the product of these binomials will become the mth power of $x+a$.

The coefficient of the second term of the product becomes equal to a multiplied by the number of factors; that is, it is equal to ma.

The coefficient of the third term reduces to a^2 repeated as many times as there are different combinations of m letters taken two and two; that is, to $\frac{m(m-1)}{1.2}a^2$.

The coefficient of the fourth term reduces to a^3 repeated as many times as there are different combinations of m letters taken three and three; that is $\frac{m(m-1)(m-2)}{1.2.3}a^3$, and so on. The last term will be a^m.

Hence the mth power of $x+a$ may be expressed as follows:

$$(x+a)^m = x^m + max^{m-1} + \frac{m(m-1)}{1.2}a^2x^{m-2} + \frac{m(m-1)(m-2)}{1.2.3}a^3x^{m-3} \dots\dots + a^m.$$

354. We perceive that *if the coefficient of any term be multiplied by the exponent of x in that term, and the product be divided by the exponent of a in that term increased by unity, it will give the coefficient of the succeeding term.*

Forming thus the seventh power of $x+a$, we obtain

$$(x+a)^7 = x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + 7a^6x + a^7.$$

We have thus deduced

Sir Isaac Newton's Binomial Theorem.

355. *In any power of a binomial $x+a$, the exponent of x begins in the first term with the exponent of the power, and in the following terms continually decreases by one. The exponent of a commences with one in the second term of the power, and continually increases by one.*

The coefficient of the first term is one, that of the second is the ex-

ponent of the power; and if the coefficient of any term be multiplied by the exponent of x in that term, and divided by the exponent of a increased by one, it will give the coefficient of the succeeding term.

356. *The coefficient of the nth term from the beginning is equal to the coefficient of the nth term from the end.*

If we change the places of x and a, we shall have, by the law of formation,

$$(a+x)^m = a^m + mxa^{m-1} + \frac{m(m-1)}{1.2}x^2a^{m-2} + \frac{m(m-1)(m-2)}{1.2.3}x^3a^{m-3} \ldots \quad + x^m.$$

The second member of this equation is the same as the second member of the equation in Art. 353, but taken in a reverse order. Comparing the two, we see that the coefficient of the second term from the beginning is equal to the coefficient of the second term from the end; the coefficient of the third from the beginning is equal to that of the third from the end, and so on. Hence, in forming any power of a binomial, it is only necessary to compute the coefficients for *half* the terms; we then repeat the same numbers in a reverse order.

357. *The mth power of $x+a$ contains $m+1$ terms.* This appears from the law of formation of the powers of a binomial developed in Art. 352. Thus the fourth power of $x+a$ contains five terms; the sixth power contains seven terms, etc.

358. *The sum of the coefficients of the terms in the nth power of $x+a$ is equal to the nth power of 2.*

For, suppose $x=1$ and $a=1$, then each term of the formula without the coefficients reduces to unity, and the sum of the terms is simply the sum of the coefficients. In this case $(x+a)^m$ becomes $(1+1)^m$, or 2^m.

Thus the coefficients of the

second power are	$1+2+1=4=2^2$,
third "	$1+3+3+1=8=2^3$,
fourth "	$1+4+6+4+1=16=2^4$, etc.

359. To obtain the development of $(x-a)^m$, it is sufficient to change $+a$ into $-a$ in the development of $(x+a)^m$. In consequence of this substitution, the terms which contain the odd powers of a will have the minus sign, while the signs of the remaining terms will be unchanged. We shall therefore have

$$(x-a)^m = x^m - max^{m-1} + \frac{m(m-1)}{1.2}a^2x^{m-2} - \frac{m(m-1)(m-2)}{1.2.3}a^3x^{m-3} + \ldots.$$

EXAMPLES.

1. Find the sixth power of $a+b$.

The terms without the coefficients are

$$a^6,\ a^5b,\ a^4b^2,\ a^3b^3,\ a^2b^4,\ ab^5,\ b^6.$$

The coefficients are

$$1,\ 6,\ \frac{6\times5}{2},\ \frac{15\times4}{3},\ \frac{20\times3}{4},\ \frac{15\times2}{5},\ \frac{6\times1}{6};$$

that is, 1, 6, 15, 20, 15, 6, 1.

Prefixing the coefficients, we obtain

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

2. Find the ninth power of $a-b$.

The terms without the coefficients are

$$a^9,\ a^8b,\ a^7b^2,\ a^6b^3,\ a^5b^4,\ a^4b^5,\ a^3b^6,\ a^2b^7,\ ab^8,\ b^9.$$

The coefficients are

$$1,\ 9,\ \frac{9\times8}{2},\ \frac{36\times7}{3},\ \frac{84\times6}{4},\ \frac{126\times5}{5},\ \frac{126\times4}{6},\ \frac{84\times3}{7},\ \frac{36\times2}{8},\ \frac{9\times1}{9};$$

that is,

1, 9, 36, 84, 126, 126, 84, 36, 9, 1.

Prefixing the coefficients, we obtain

$$(a-b)^9 = a^9 - 9a^8b + 36a^7b^2 - 84a^6b^3 + 126a^5b^4 - 126a^4b^5 + 84a^3b^6 - 36a^2b^7 + 9ab^8 - b^9.$$

It should be remembered that it is only necessary to compute the coefficients of *half* the terms independently.

3. Find the seventh power of $a-x$.
4. Find the third term of $(a+b)^{15}$.
5. Find the forty-ninth term of $(a-x)^{50}$.
6. Find the middle term of $(a+x)^{10}$.

360. *A Binomial with Coefficients.*—If the terms of the given binomial have coefficients or exponents, we may obtain any power of it by means of the binomial formula. For this purpose, each term must be raised to its proper power denoted by the exponents in the binomial formula.

7. Find the fourth power of $2x+3a$.

For convenience, let us substitute y for $2x$ and b for $3a$.

Then $$(y+b)^4=y^4+4y^3b+6y^2b^2+4yb^3+b^4.$$

Restoring the values of y and b,

the first term will be $(2x)^4=16x^4$,

the second term will be $4(2x)^3\times 3a=96x^3a$,

the third term will be $6(2x)^2\times(3a)^2=216x^2a^2$,

the fourth term will be $4(2x)\times(3a)^3=216xa^3$,

the fifth term will be $(3a)^4=81a^4$.

Therefore $(2x+3a)^4=16x^4+96x^3a+216x^2a^2+216xa^3+81a^4$.

It is recommended to write the three factors of each term in a vertical column, and then perform the multiplication as indicated below:

Coefficients,	1	$+\ 4$	$+\ 6$	$+\ 4$	$+\ 1$
Powers of $2x$,	$16x^4$	$+\ 8x^3$	$+\ 4x^2$	$+\ 2x$	$+\ 1$
Powers of $3a$,	1	$+\ 3a$	$+\ 9a^2$	$+\ 27a^3$	$+81a^4$

$$(2x+3a)^4=16x^4+96x^3a+216x^2a^2+216xa^3+81a^4.$$

8. Find the fifth power of $2ax-3b$.

Coefficients,	1	$+\ 5$	$+\ 10$	$+\ 10$	$+\ 5$	$+\ 1$
Powers of $2ax$,	$32a^5x^5$	$+\ 16a^4x^4$	$+\ 8a^3x^3$	$+\ 4a^2x^2$	$+\ 2ax$	$+\ 1$
Powers of $-3b$,	1	$-\ 3b$	$+\ 9b^2$	$-\ 27b^3$	$+\ 81b^4$	$-243b^5$

$$(2ax-3b)^5=32a^5x^5-240a^4x^4b+720a^3x^3b^2-1080a^2x^2b^3+810axb^4-243b^5.$$

9. Find the fourth power of $2x+5a^2$.

Ans. $16x^4+160x^3a^2+600x^2a^4+1000xa^6+625a^8$.

10. Find the fourth power of x^3+4y^2.

11. Find the sixth power of a^3+3ab.

Ans. $a^{18}+18a^{16}b+135a^{14}b^2+540a^{12}b^3$
$+1215a^{10}b^4+1458a^8b^5+729a^6b^6$.

12. Find the seventh power of $2a-3b$.

Ans. $128a^7-1344a^6b+6048a^5b^2-15120a^4b^3$
$+22680a^3b^4-20412a^2b^5+10206ab^6-2187b^7$.

13. Find the fifth power of $5a^2-4x^2y$.

14. Find the sixth power of a^2x+by^2.

15. Find the fifth power of $ax-1$.

16. Find the fifth term of $(a^2-b^2)^{12}$.

17. Find the fifth term of $(3x^{\frac{1}{2}}-4y^{\frac{1}{2}})^9$.

18. Find the sixth power of $5-\frac{x}{6}$.

Powers and Roots of Polynomials.

361. If it is required to raise a *polynomial* to any power, we may, by substituting other letters, reduce it to the form of a binomial. We obtain the power of this binomial by the general formula; then, restoring the original letters, and performing the operations indicated, we obtain the required power of the proposed polynomial.

Ex. 1. Let it be required to raise $a+b+c$ to the third power.

If we put $b+c=m$, we shall have

$$(a+b+c)^3=(a+m)^3=a^3+3a^2m+3am^2+m^3,$$

or

$$a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3.$$

Developing the powers of the binomial $b+c$, and performing the operations indicated, we obtain

$$(a+b+c)^3=a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3.$$

Ex. 2. Find the fifth power of $x+a+b$.

Ex. 3. Find the fourth power of a^2-ab+b^2.

Ans. $a^8-4a^7b+10a^6b^2-16a^5b^3+19a^4b^4-16a^3b^5+10a^2b^6-4ab^7+b^8$.

Ex. 4. Find the fifth power of $1+2x+3x^2$.

Ex. 5. Find the sixth power of $a+b+c$.

Ans. $a^6+6a^5b+6a^5c+15a^4b^2+30a^4bc+15a^4c^2+20a^3b^3+60a^3b^2c+60a^3bc^2+20a^3c^3+15a^2b^4+60a^2b^3c+90a^2b^2c^2+60a^2bc^3+15a^2c^4+6ab^5+30ab^4c+60ab^3c^2+60ab^2c^3+30abc^4+6ac^5+b^6+6b^5c+15b^4c^2+20b^3c^3+15b^2c^4+6bc^5+c^6$.

362. The binomial theorem will inform us how to extract *any root of a polynomial.* We know that the mth power of

$x+a$ is $x^m+max^{m-1}+$other terms. The first term of the root is, therefore, the mth root of the first term of the polynomial. Also the second term of the root may be found by dividing the second term of the polynomial by mx^{m-1}; that is, the first term of the root raised to the next inferior power, and multiplied by the exponent of the given power. Hence, for extracting any root of a polynomial, we have the following

RULE.

Arrange the terms according to the powers of one of the letters, and take the mth root of the first term for the first term of the required root.

Subtract the mth power of this term of the root from the given polynomial, and divide the first term of the remainder by m times the $(m-1)$ *power of this root; the quotient will be the second term of the root.*

Subtract the mth power of the terms already found from the given polynomial, and, using the same divisor, proceed in like manner to find the remaining terms of the root.

Ex. 1. Find the fourth root of

$$16a^4-96a^3x+216a^2x^2-216ax^3+81x^4.$$

$$\begin{array}{rl} & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4\ (2a-3x \\ & 16a^4 \\ 32a^3) & \quad -96a^3x \\ & 16a^4-96a^3x+216a^2x^2-216ax^3+81x^4. \end{array}$$

Here we take the fourth root of $16a^4$, which is $2a$, for the first term of the required root, subtract its fourth power, and bring down the first term of the remainder, $-96a^3x$. For a divisor, we raise the first term of the root to the third power and multiply it by 4, making $32a^3$. Dividing, we obtain $-3x$ for the second term of the root. The quantity $2a-3x$, being raised to the fourth power, is found to be equal to the proposed polynomial.

Ex. 2. Find the fifth root of

$$80x^3+32x^5-80x^4-40x^2+10x-1.$$

Ans. $2x-1$.

Ex. 3. Find the fourth root of

$$336x^5+81x^8-216x^7-56x^4+16-224x^3+64x.$$

Ans. $3x^2-2x-2$.

363. *To extract any Root of a Number.*—The preceding method may be applied to the extraction of any root of a number. Let n be the index of the root, n being any whole number. For a reason similar to that given for the square and cube roots, we must first divide the number into periods of n figures each, beginning at the right. The left-hand period may contain less than n figures. Then the first figure of the required root will be the nth root of the greatest nth power contained in the first period on the left. If we subtract the nth power of this root from the given number, and divide the remainder by n times the $(n-1)$th power of the first figure, regarding its local value, the quotient will be the second figure of the root, or possibly a figure too large. The result may be tested by raising the whole root now found to the nth power; and if there are other figures they may be found in the same manner.

In the extraction of the nth root of an integer, if there is still a remainder after we have obtained the units' figure of the root, it indicates that the proposed number has not an exact nth root. We may, if we please, proceed with the approximation to any desired extent by annexing any number of periods of n ciphers each, and continuing the operation. We thus obtain a decimal part to be added to the integral part already found.

So, also, if a decimal number has no exact nth root, we may annex ciphers, and proceed with the approximation to any desired extent, dividing the number into periods commencing with the decimal point.

Ex. 1. Find the fifth root of 33554432.

$$\begin{array}{rl} & 335.54432\,(32 \\ & 243 \\ \hline 5.3^4=405\,) & 925 \\ 32^5= & 33554432. \end{array}$$

Ex. 2. Find the fifth root of 4984209207.

Ex. 3. Find the fifth root of 10.

Ex. 4. Find the fifth root of $\frac{2}{3}$. *Ans.* .922.

364. When the index of the required root is composed of *two factors*, we may obtain the root required by the successive extraction of simpler roots, Art. 217. For the *mn*th root of any number is equal to the *m*th root of the *n*th root of that number.

Thus we may obtain the *fourth* root by extracting the square root of the square root.

We may obtain the *sixth* root by extracting the cube root of the square root, or the square root of the cube root. It is, however, best to extract the roots of the lowest degrees first, because the operation is less laborious.

We may obtain the *eighth* root by extracting the square root *three times* successively. We may obtain the *ninth* root by extracting the cube root *twice* successively.

Ex. 1. Find the fourth root of

$$6a^2b^2+a^4-4a^3b-4ab^3+b^4.$$

Ex. 2. Find the sixth root of

$$6a^5b+15a^4b^2+a^6+20a^3b^3+15a^2b^4+b^6+6ab^5.$$

Ex. 3. Find the eighth root of

$$1024x^7y+1792x^6y^2+256x^8+1120x^4y^4+1792x^5y^3+448x^3y^5+y^8+112x^2y^6+16xy^7.$$

CHAPTER XIX.

SERIES.

365. A *series* is a succession of terms each of which is derived from one or more of the preceding ones by a fixed law. This law is called the *law of the series.* The number of terms of the series is generally unlimited. Arithmetical and geometrical progressions afford examples of series.

366. A *converging* series is one in which the sum of the first n terms can not numerically exceed some finite quantity, however great n may be.

Thus, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc., is a converging series.

367. A *diverging* series is one in which n can be taken so large that the sum of the first n terms is numerically greater than any finite quantity.

Thus, 1, 2, 3, 4, 5, 6, etc., is a diverging series.

368. When a certain number of terms are given, and the law of the series is known, we may find any term of the series, or the sum of any number of terms. This may generally be done by the *method of differences.*

369. *To find the several orders of differences for any series:*

Subtract the first term from the second, the second from the third, the third from the fourth, etc.; we shall thus form a new series, which is called the *first order of differences.*

Subtract the first term of this new series from the second, the second from the third, etc.; we shall thus form a third series, called the *second order of differences.*

Proceed in like manner for the third, fourth, etc., orders of differences, and so on till they terminate, or are carried as far as may be thought necessary.

Ex. 1. Find the several orders of differences of the series of square numbers 1, 4, 9, 16, etc.

Squares.	1st Diff.	2d Diff.	3d Diff.
1			
	3		
4		2	
	5		0
9		2	
	7		0
16		2	
	9		0
25		2	

Ex. 2. Find the several orders of differences of the series of cube numbers 1, 8, 27, etc.

Cubes.	1st Diff.	2d Diff.	3d Diff.	4th Diff.
1				
	7			
8		12		
	19		6	
27		18		0
	37		6	
64		24		0
	61		6	
125		30		0
	91		6	
216		36		0

Ex. 3. Find the several orders of differences of the series of fourth powers 1, 16, 81, 256, 625, 1296, etc.

Ex. 4. Find the several orders of differences of the series of fifth powers 1, 32, 243, 1024, 3125, 7776, 16807, etc.

Ex. 5. Find the several orders of differences of the series of numbers 1, 3, 6, 10, 15, 21, etc.

370. *To find the nth term of any series:*

Let a, b, c, d, e, etc., represent the proposed series. If we subtract each term from the next succeeding one, we shall obtain the first order of differences; if we subtract each term of this new series from the succeeding term, we shall obtain the second order of differences, and so on, as exhibited in the following table:

Series.	1st Diff.	2d Differences.	3d Order of Differences.	4th Order of Differences.
a				
	$b-a$			
b		$c-2b+a$		
	$c-b$		$d-3c+3b-a$	
c		$d-2c+b$		$e-4d+6c-4b+a$
	$d-c$		$e-3d+3c-b$	
d		$e-2d+c$		
	$e-d$			
e				

Let D′, D″, D‴, D⁗, etc., represent the first terms of the several orders of differences. Then we shall have

$D'=b-a,$ whence $b=a+D'.$

$D''=c-2b+a,$ " $c=a+2D'+D''.$

$D'''=d-3c+3b-a,$ " $d=a+3D'+3D''+D'''.$

$D''''=e-4d+6c-4b+a,$ " $e=a+4D'+6D''+4D'''+D''''$,

etc., etc.

The coefficients of the value of c, the *third* term of the proposed series, are 1, 2, 1, which are the coefficients of the *second* power of a binomial; the coefficients of the value of d, the fourth term, are 1, 3, 3, 1, which are the coefficients of the *third* power of a binomial, and so on. Hence we infer that the coefficients of the nth term of the series are the coefficients of the $(n-1)$th power of a binomial. If we denote the nth term of the series by T_n, we shall have

$$T_n=a+(n-1)D'+\frac{(n-1)(n-2)}{2}D''+\frac{(n-1)(n-2)(n-3)}{2.3}D''' +, \text{ etc.}$$

Ex. 1. Find the 12th term of the series 2, 6, 12, 20, 30, etc.

The first order of differences, 4, 6, 8, 10, etc.

" second order of differences, 2, 2, 2, etc.

" third order of differences, 0, 0.

Here $D'=4$, $D''=2$, and $D'''=0$. Also $a=2$ and $n=12$.

Hence $T_{12}=2+11D'+55D''=2+44+110=156$, *Ans.*

Ex. 2. Find the twentieth term of the series

1, 3, 6, 10, 15, 21, etc.

Here $D'=2$, $D''=1$, $a=1$, and $n=20$.

Therefore $T_{20}=1+19D'+171D''=1+38+171=210$, *Ans.*

Ex. 3. Find the thirteenth term of the series

1, 5, 14, 30, 55, 91, etc.

Ex. 4. Find the fifteenth term of the series

1, 4, 9, 16, 25, 36, etc.

Ex. 5. Find the twentieth term of the series

1, 8, 27, 64, 125, etc.

Ex. 6. Find the nth term of the series 1, 3, 6, 10, 15, 21, etc.

Ans. $\frac{n(n+1)}{2}$.

Ex. 7. Find the nth term of the series 1, 4, 10, 20, 35, etc.

Ans. $\frac{n(n+1)(n+2)}{6}$.

Ex. 8. Find the nth term of the series 1, 5, 15, 35, 70, 126, etc.

Ans. $\frac{n(n+1)(n+2)(n+3)}{24}$.

371. *To find the sum of n terms of any series:*

Let us assume the series

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d, \text{ etc.} \qquad (1.)$$

Subtracting each term from the next succeeding, we obtain the first order of differences,

$$a,\ b,\ c,\ d, \text{ etc.} \qquad (2.)$$

Now it is clear that the sum of n terms of the series (2) is equal to the $(n+1)$th term of series (1); and the nth order of differences in series (2) is the $(n+1)$th order in series (1). If, then, we denote the sum of n terms of series (2) by S, which is the same as the $(n+1)$th term of (1), we may obtain the value of S from the formula of the preceding article by substituting

0 for a,
$n+1$ for n,
a for D′,
D′ for D″, etc.

Hence

$$S=na+\frac{n(n-1)}{2}D'+\frac{n(n-1)(n-2)}{2.3}D''+\frac{n(n-1)(n-2)(n-3)}{2.3.4}D''', \text{ etc.}$$

When any one of the successive orders of differences becomes zero, this formula gives the exact sum of the terms. When no order of differences becomes zero, the formula may still give approximate results, which will, in general, be nearer the truth the greater the number of terms employed.

EXAMPLES.

1. Find the sum of 15 terms of the series

1, 3, 6, 10, 15, 21, etc.

Here $a=1$, D′$=2$, D″$=1$, D‴$=0$.

Therefore $S=15+15.14+5.7.13=680$, *Ans.*

2. Find the sum of 20 terms of the series

$$1, 4, 10, 20, 35, \text{etc.}$$

3. Find the sum of n terms of the series

$$1, 2, 3, 4, 5, 6, \text{etc.}$$

Ans. $\frac{n(n+1)}{2}$.

4. Find the sum of n terms of the series

$$1^2, 2^2, 3^2, 4^2, 5^2, \text{etc.}$$

Ans. $\frac{n(n+1)(2n+1)}{6}$.

5. Find the sum of n terms of the series

$$1^3, 2^3, 3^3, 4^3, 5^3, \text{etc.}$$

Ans. $\frac{(n^2+n)^2}{4}$.

6. Find the sum of n terms of the series

$$1, 3, 6, 10, 15, \text{etc.}$$

Ans. $\frac{n(n+1)(n+2)}{2.3}$.

7. Find the sum of n terms of the series

$$1.2,\ 2.3,\ 3.4,\ 4.5,\ 5.6, \text{etc.}$$

Ans. $\frac{n(n+1)(n+2)}{3}$.

8. Find the sum of n terms of the series

$$1, 4, 10, 20, 35, \text{etc.}$$

Ans. $\frac{n(n+1)(n+2)(n+3)}{2.3.4}$.

Interpolation.

372. *Interpolation* is the process by which, when we have given a certain number of terms of a series, we compute intermediate terms which conform to the law of the series.

Interpolation may, in most cases, be effected by the use of the formula of Art. 370. If in this formula we substitute $n+1$ for n, we shall have

$$T_{n+1}=a+nD'+\frac{n(n-1)}{2}D''+\frac{n(n-1)(n-2)}{2.3}D'''+, \text{etc.},$$

which expresses the value of that term of the series which has n terms before it. When n is a *fraction less than unity*, T_{n+1}

stands for a term between the first and second of the given terms. When n is greater than 1 and less than 2, the intermediate term will lie between the second and third of the given terms, and so on. In general, the preceding formula will give the value of such intermediate terms.

EXAMPLES.

1. Given the cube root of 60 equal to 3.914868,

"	"	"	61	"	3.936497,
"	"	"	62	"	3.957891,
"	"	"	63	"	3.979057,
"	"	"	64	"	4.000000,

to find the cube root of 60.25.

Here $D'=+.021629$, $D''=-.000235$, $D'''=+.000007$, etc.

$$a=3.914868, \text{ and } n=.25.$$

Substituting the value of n in the formula, we have

$$T_{n+1}=a+\tfrac{1}{4}D'-\tfrac{3}{32}D''+\tfrac{7}{128}D'''-, \text{ etc.}$$

The value of the		1st term	is	+3.914868,
"	"	2d	"	+ .005407,
"	"	3d	"	+ .000022,
"	"	4th	"	+ .000000.
Hence the cube root of 60.25 is				3.920297.

2. Find the cube root of 60.5. *Ans.* 3.925712.
3. Find the cube root of 60.75. *Ans.* 3.931112.
4. Find the cube root of 60.6. *Ans.* 3.927874.
5. Find the cube root of 60.33. *Ans.* 3.922031.
6. Given the square root of 30 equal to 5.477226,

"	"	"	31	"	5.567764,
"	"	"	32	"	5.656854,
"	"	"	33	"	5.744563,
"	"	"	34	"	5.830952,

to find the square root of 30.3 *Ans.* 5.504544.

7. Find the square root of 30.4. *Ans.* 5.513619.
8. Find the square root of 30.5. *Ans.* 5.522681.
9. Find the square root of 30.6. *Ans.* 5.531727.
10. Find the square root of 30.8. *Ans.* 5.549775.

Development of Algebraic Expressions into Series.

373. An irreducible fraction may be converted into an infinite series by dividing the numerator by the denominator, according to the usual method of division.

Ex. 1. Expand $\frac{1}{1-x}$ into an infinite series.

$$\begin{array}{l} 1-x)1 \quad (1+x+x^2+x^3+x^4+, \text{ etc.} \\ \underline{1-x} \\ \quad x \\ \quad \underline{x-x^2} \\ \qquad x^2 \\ \qquad \underline{x^2-x^3} \\ \qquad\quad x^3 \\ \qquad\quad \underline{x^3-x^4} \\ \qquad\qquad x^4 \end{array}$$

Hence $\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+$, etc., to infinity.

Suppose $x=\frac{1}{2}$, we shall then have

$$\frac{1}{1-x}=\frac{1}{1-\frac{1}{2}}=2=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+, \text{ etc.}$$

Suppose $x=\frac{1}{3}$, we shall then have

$$\frac{1}{1-x}=\frac{1}{1-\frac{1}{3}}=\tfrac{3}{2}=1+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{81}+, \text{ etc.}$$

Ex. 2. Convert $\frac{1}{1+x}$ into an infinite series.

Ans. $1-x+x^2-x^3+x^4-x^5+$, etc.

Suppose $x=\frac{1}{2}$, we shall then have

$$\frac{1}{1+\frac{1}{2}}=\tfrac{2}{3}=1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{16}-\tfrac{1}{32}+, \text{ etc.}$$

Ex. 3. Convert $\frac{a}{a+x}$ into an infinite series.

Ans. $1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-$, etc.

Ex. 4. Convert $\frac{a}{a-x}$ into an infinite series.

Ans. $1+\frac{x}{a}+\frac{x^2}{a^2}+\frac{x^3}{a^3}+\frac{x^4}{a^4}+$, etc.

Ex. 5. Convert $\frac{1+x}{1-x}$ into an infinite series.

Ex. 6. Convert $\frac{a+x}{a-x}$ into an infinite series.

Ans. $1+\frac{2x}{a}+\frac{2x^2}{a^2}+\frac{2x^3}{a^3}+\frac{2x^4}{a^4}+$, etc.

Ex. 7. Convert $\frac{1}{1-x+x^2}$ into an infinite series.

Ans. $1+x-x^3-x^4+x^6+x^7-$, etc.

Ex. 8. Convert $\frac{1-x}{1-x+x^2}$ into an infinite series.

Ans. $1-x^2-x^3+x^5+x^6-x^8-$, etc.

Ex. 9. Convert $\frac{1+x}{1-x-x^2}$ into an infinite series.

Ans. $1+2x+3x^2+5x^3+8x^4+13x^5+$, etc.

374. An algebraic expression which is not a perfect square may be developed into an infinite series by extracting its square root according to the method of Art. 198.

Ex. 1. Develop the square root of $1+x$ into an infinite series.

$$1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\text{, etc.}\right.$$

$$\begin{array}{rl}
 & 1 \\
\hline
2+\frac{x}{2}) & x \\
 & x+\frac{x^2}{4} \\
\hline
2+x-\frac{x^2}{8}) & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\hline
2+x-\frac{x^2}{4}+\frac{x^3}{16}) & \frac{x^3}{8}-\frac{x^4}{64} \\
 & \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\hline
 & -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}.
\end{array}$$

Hence the square root of $1+x$ is equal to

$$1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+, \text{ etc.}$$

Suppose $x=1$, we shall have

$$\sqrt{2}=1+\tfrac{1}{2}-\tfrac{1}{8}+\tfrac{1}{16}-\tfrac{5}{128}+, \text{ etc.}$$

Ex. 2. Develop the square root of a^2+x into an infinite series.

$$\textit{Ans. } a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+, \text{ etc.}$$

Ex. 3. Develop the square root of a^4+x into an infinite series.

Ex. 4. Develop the square root of a^4-x into an infinite series.

Ex. 5. Develop the square root of a^2+x^2 into an infinite series.

Method of Undetermined Coefficients.

375. One of the most useful methods of developing algebraic expressions into series is the method of *undetermined coefficients.* It consists in assuming the required development in the form of a series with unknown coefficients, and afterward finding the value of these coefficients. This method is founded upon the properties of identical equations.

376. An *identical equation* is one in which the two members are identical, or may be reduced to identity by performing the operations indicated in them. As

$$ax+b=ax+b,$$

$$\frac{a^2-x^2}{a-x}=a+x,$$

$$a-\frac{a}{1+x}=\frac{ax}{1+x}.$$

377. It follows from the definition that *an identical equation is satisfied by each and every value which may be assigned to a letter which it contains,* provided that value is the same in both members of the equation.

Every identical equation containing but one unknown quantity can be reduced to the form of

$$A+Bx+Cx^2+Dx^3+, \text{ etc.}=A'+B'x+C'x^2+D'x^3+, \text{ etc.}$$

378. *If an equation of the form*

$$A+Bx+Cx^2+, \text{ etc.}=A'+B'x+C'x^2+, \text{ etc.},$$

must be satisfied for each and every value given to x, then the coefficients of the like powers of x in the two members are equal each to each.

For, since this equation must be satisfied for every value of x, it must be satisfied when $x=0$. But upon this supposition all the terms vanish except two, and we have

$$A=A'.$$

Suppressing these two equal terms, we have

$$Bx+Cx^2+, \text{ etc.}=B'x+C'x^2+, \text{ etc.}$$

Dividing each term by x, we obtain

$$B+Cx+, \text{ etc.}=B'+C'x+, \text{ etc.}$$

Since this equation must be satisfied for every value of x, it must be satisfied when $x=0$. But upon this supposition

$$B=B'.$$

In the same manner we can prove that

$$C=C',$$
$$D=D', \text{ etc.}$$

379. *Whenever we have an equation of the form*

$$M+Nx+Px^2+Qx^3+, \text{ etc.}=0,$$

which is true for every value of x, all the coefficients of x are equal to zero.

For, if we transpose all the terms of the equation in the last article to the left-hand member, we shall have

$$A-A'+(B-B')x+(C-C')x^2+(D-D')x^3+, \text{ etc.}=0.$$

But it has been shown that $A=A'$, $B=B'$, etc.; whence $A-A'=0$, $B-B'=0$, etc. If we substitute M for $A-A'$, and N for $B-B'$, etc., the equation will be

$$M+Nx+Px^2+Qx^3+, \text{ etc.}=0.$$

whence $M=0,\ N=0,\ P=0$, etc.

Ex. 1. Expand the fraction $\dfrac{1+2x}{1-3x}$ into an infinite series.

It is plain that this development is possible, for we may divide the numerator by the denominator, as explained in Art. 373.

Let us, then, assume the identical equation

$$\frac{1+2x}{1-3x}=A+Bx+Cx^2+Dx^3+Ex^4+, \text{ etc.},$$

where the coefficients A, B, C, D are supposed to be independent of x, but dependent on the known terms of the fraction.

In order to obtain the values of these coefficients, let us clear this equation of fractions, and we shall have

$$1+2x=A+(B-3A)x+(C-3B)x^2+(D-3C)x^3+$$
$$(E-3D)x^4+, \text{ etc.}$$

Now, since this is supposed to be an identical equation, the coefficients of the like powers of x in the two members are equal each to each.

Therefore $A=1$.

$B-3A=2$, whence $B=5$;
$C-3B=0$, " $C=15$;
$D-3C=0$, " $D=45$;
$E-3D=0$, " $E=135$, etc.

Substituting these values of the coefficients in the assumed series, we obtain

$$\frac{1+2x}{1-3x}=1+5x+15x^2+45x^3+135x^4+, \text{ etc.},$$

where the coefficient of each term after the second is three times the coefficient of the preceding term.

380. The method thus exemplified is expressed in the following

RULE.

Assume the proposed expression equal to a series of the form $A+Bx+Cx^2+$, etc.; *clear the equation of fractions, or raise it to its proper power, and place the coefficients of the like powers of* x *in the two members equal each to each. Then find from these equations the values of* A, B, C, etc., *and substitute these values in the assumed development.*

Ex. 2. Expand the fraction $\frac{1}{1-2x+x^2}$ into an infinite series.

Assume $\frac{1}{1-2x+x^2}=A+Bx+Cx^2+Dx^3+Ex^4+$, etc.

Clearing of fractions, we have

$$1=A+(B-2A)x+(C-2B+A)x^2+(D-2C+B)x^3$$
$$+(E-2D+C)x^4+, \text{ etc.}$$

Therefore we must have

$$\begin{aligned}
A&=1, & & \\
B-2A&=0, &\text{whence } & B=2A=2; \\
C-2B+A&=0, &\text{"}\quad & C=2B-A=3; \\
D-2C+B&=0, &\text{"}\quad & D=2C-B=4; \\
E-2D+C&=0, &\text{"}\quad & E=2D-C=5, \text{ etc.}
\end{aligned}$$

Therefore $\dfrac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+$, etc.

Ex. 3. Expand the fraction $\dfrac{1+2x}{1-x-x^2}$ into an infinite series.

Ans. $1+3x+4x^2+7x^3+11x^4+18x^5+29x^6+$, etc., where the coefficient of each term is equal to the sum of the coefficients of the two preceding terms.

Ex. 4. Expand $\dfrac{1-x}{1-2x-3x^2}$ into an infinite series.

Ans. $1+x+5x^2+13x^3+41x^4+121x^5+$, etc.

What is the law of the coefficients in this series?

Ex. 5. Expand $\sqrt{1-x}$ into an infinite series.

Ans. $1-\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{16}-\dfrac{5x^4}{128}-\dfrac{7x^5}{256}-$, etc.

Ex. 6. Expand $\dfrac{1-x}{1+x+x^2}$ into an infinite series.

Ans. $1-2x+x^2+x^3-2x^4+x^5+x^6-$, etc.

Ex. 7. Expand $\sqrt{a^2-x^2}$ into an infinite series.

381. *Proper Form of the assumed Series.*—In applying the method of undetermined coefficients to develop algebraic expressions into series, we should determine what power of the variable will be contained in the first term of the development, and assume a corresponding series of terms. Generally the first term of the development is constant, or contains x^0; but the first term of the series may contain x with any exponent either positive or negative. If the assumed development commences with a power of x *lower* than is necessary, no error will result, for the coefficients of the redundant terms will reduce to zero. But if the assumed development commences with a power of x *higher* than it should, the fact will be indicated by an *absurdity* in one of the resulting equations.

The form of the series which should be adopted in each case may be determined by putting $x=0$, and observing the nature of the result. If in this case the proposed expression becomes equal to a finite quantity, the first term of the series will not contain x. If the expression reduces to zero, the first term will contain x; and if the expression reduces to the form $\frac{A}{0}$, then the first term of the development must contain x with a negative exponent.

Let it be required to develop $\frac{1}{3x-x^2}$ into a series.

Assume $$\frac{1}{3x-x^2}=A+Bx+Cx^2+Dx^3+, \text{ etc.}$$

Clearing of fractions, we have

$$1=3Ax+(3B-A)x^2+, \text{ etc.},$$

whence, according to Art. 378, we obtain $1=0$, which is absurd, and shows that the assumed form is not applicable in the present case.

Let us, however, assume

$$\frac{1}{3x-x^2}=Ax^{-1}+B+Cx+Dx^2+, \text{ etc.}$$

Clearing of fractions, we have

$$1=3A+(3B-A)x+(3C-B)x^2+(3D-C)x^3+, \text{ etc.}$$

Therefore $3A=1$, whence $A=\frac{1}{3}$;

$3B-A=0$, " $B=\frac{1}{9}$;

$3C-B=0$, " $C=\frac{1}{27}$;

$3D-C=0$, " $D=\frac{1}{81}$.

Substituting these values, we find

$$\frac{1}{3x-x^2}=\frac{x^{-1}}{3}+\frac{x^0}{9}+\frac{x}{27}+\frac{x^2}{81}+, \text{ etc.}$$

To Resolve a Fraction into Simpler Fractions.

382. When the denominator of a fraction can be resolved into factors, the principles now developed enable us to resolve the fraction itself into *two or more simpler fractions*, having these factors for denominators. In such a case, the given fraction is the sum of the partial fractions.

Ex. 1. Resolve the fraction $\frac{5x-12}{x^2-5x+6}$ into partial fractions.

We perceive that $x^2-5x+6=(x-2)(x-3)$.

Assume $$\frac{5x-12}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3},$$

in which the values of A and B are to be determined.

Clearing of fractions, we have

$$5x-12=(A+B)x-(3A+2B).$$

By the principle of Art. 378, $A+B=5$,

and $$3A+2B=12.$$

From which we obtain $A=2$ and $B=3$.

Substituting in the assumed equation, we have

$$\frac{5x-12}{x^2-5x+6}=\frac{2}{x-2}+\frac{3}{x-3}.$$

Ex. 2. Resolve $\frac{5x+1}{x^2-1}$ into partial fractions.

Ans. $\frac{2}{x+1}+\frac{3}{x-1}$.

Ex. 3. Resolve $\frac{5x-19}{x^2-8x+15}$ into partial fractions.

Ans. $\frac{3}{x-5}+\frac{2}{x-3}$.

Ex. 4. Resolve $\frac{3x^2-1}{x^3-x}$ into partial fractions.

Ans. $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{x}$.

Ex. 5. Resolve $\frac{2x^2-6x+6}{(x-1)(x-2)(x-3)}$ into partial fractions.

Ans. $\frac{1}{x-1}-\frac{2}{x-2}+\frac{3}{x-3}$.

Ex. 6. Resolve $\frac{5x^2+2x-1}{(x+1)(x-1)(2x+1)}$ into partial fractions.

Ans. $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{2x+1}$.

Ex. 7. Resolve $\frac{13+21x+2x^2}{1-5x^2+4x^4}$ into partial fractions.

Ans. $\frac{1}{1+x}-\frac{6}{1-x}+\frac{2}{1+2x}+\frac{16}{1-2x}$.

Reversion of Series.

383. *The reversion of a series* is the finding the value of the unknown quantity contained in an infinite series by means of another series involving the powers of some other quantity.

This may be accomplished by the method of undetermined coefficients in a mode similar to that employed in Art. 379.

Ex. 1. Given the series $y=x+x^2+x^3+$, etc., to find the value of x in terms of y.

Assume $\qquad x=Ay+By^2+Cy^3+Dy^4+$, etc.

Find, by involution, the values of x^2, x^3, x^4, and x^5, carrying each result only to the term containing y^5. Then, substituting these values for x, x^2, x^3, etc., in the given equation, we shall have

$$
y = Ay + \left.\begin{array}{l} B \\ +A^2 \end{array}\right| y^2 + \left.\begin{array}{l} C \\ +2AB \\ +A^3 \end{array}\right| y^3 + \left.\begin{array}{l} D \\ +2AC \\ +B^2 \\ +3A^2B \\ +A^4 \end{array}\right| y^4 + \left.\begin{array}{l} E \\ +2AD \\ +2BC \\ +3A^2C \\ +3AB^2 \\ +4A^3B \\ +A^5 \end{array}\right| y^5 + \text{, etc.}
$$

Since this is an identical equation, we place the coefficients of the like powers of y in the two members equal to each other, and we obtain

$$A=+1,\ B=-1,\ C=+1,\ D=-1,\ E=+1, \text{ etc.}$$

Hence we have $x=y-y^2+y^3-y^4+y^5-$, etc., *Ans.*

Ex. 2. Given the series $y=x-\frac{x^2}{2}+\frac{x^3}{4}-\frac{x^4}{8}+$, etc., to find the value of x in terms of y.

Ans. $x=y+\frac{y^2}{2}+\frac{y^3}{4}+\frac{y^4}{8}+$, etc.

Ex. 3. Given the series $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-$, etc., to find the value of x in terms of y.

Ans. $x=y+\frac{y^2}{2}+\frac{y^3}{2.3}+\frac{y^4}{2.3.4}+\frac{y^5}{2.3.4.5}+$, etc.

Ex. 4. Given the series $y=x+x^3+x^5+x^7+x^9+$, etc., to find the value of x in terms of y.

Ans. $x=y-y^3+2y^5-5y^7+14y^9-$, etc.

Ex. 5. Given the series $y=x+3x^2+5x^3+7x^4+9x^5+$, etc., to find the value of x in terms of y.

Ans. $x=y-3y^2+13y^3-67y^4+381y^5-$, etc.

384. When the sum of a series is known, we may sometimes obtain the approximate *value of the unknown quantity* by reverting the series.

Ex. 1. Given $\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+$, etc. $=\frac{1}{4}$, to find the value of x.

If we call s the sum of the series, and proceed as in the last article, we shall have

$$x=2s-s^2+\tfrac{2}{3}s^3-\tfrac{1}{2}s^4+\tfrac{2}{5}s^5-, \text{ etc.}$$

Substituting the value of s, we find

$x=\frac{1}{2}-\frac{1}{16}+\frac{1}{96}-\frac{1}{512}+\frac{1}{2560}-$, etc., or $x=0.446354$ nearly.

Ex. 2. Given $2x+3x^3+4x^5+5x^7+$, etc. $=\frac{1}{2}$, to find the value of x.

Ans. $x=\frac{s}{2}-\frac{3s^3}{16}+\frac{19s^5}{128}-\frac{152s^7}{1024}+$, etc.,

or $x=\frac{1}{4}-\frac{3}{128}+\frac{19}{4096}-\frac{19}{16384}+$, etc. $=0.2300$ nearly.

Ex. 3. Given $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+$, etc. $=\frac{1}{3}$, to find the value of x.

Ans. $x=s+\frac{s^3}{3}+\frac{2s^5}{15}+\frac{17s^7}{351}+$, etc.,

or $x=\frac{1}{3}+\frac{1}{81}+\frac{2}{3645}+\frac{17}{767637}+$, etc. $=.34625$ nearly.

Ex. 4. Given $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+$, etc. $=\frac{1}{5}$, to find the value of x.

Ans. $x=s-\frac{s^2}{2}+\frac{s^3}{6}-\frac{s^4}{24}+\frac{s^5}{120}-\frac{s^6}{720}+$, etc.,

or $x=\frac{1}{5}-\frac{1}{50}+\frac{1}{750}-\frac{1}{15000}+\frac{1}{375000}-\frac{1}{11250000}+$, etc.,

or $x=0.1812692$ nearly.

Binomial Theorem.

385. In Art. 353 the binomial theorem was demonstrated for the case in which m is a positive whole number. By means of the method of undetermined coefficients we can prove that this formula is true, whether m is *positive or negative, entire or fractional.* The demonstration of this theorem depends upon the following proposition:

386. *The value of* $\dfrac{a^n-b^n}{a-b}$, *when* $a=b$, *is in all cases* na^{n-1}, *whether n is positive or negative, integral or fractional.*

First. It was shown in Art. 83 that when n is a positive whole number, a^n-b^n is exactly divisible by $a-b$, and the quotient is $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots+b^{n-1}$. The number of terms in this quotient is equal to n; for b is contained in all the terms except the first, and the exponents of b are 1, 2, 3, etc., to $n-1$, so that the number of terms containing b is $n-1$, and the whole number of terms is equal to n. Now, when $a=b$, each term of the above quotient becomes a^{n-1}, and, since there are n terms in the quotient, this quotient reduces to na^{n-1}.

Second. Suppose n to be *a positive fraction*, or $n=\dfrac{p}{q}$, where p and q are positive whole numbers.

Let $\quad a^{\frac{1}{q}}=x$, whence $a^{\frac{p}{q}}=x^p$, and $a=x^q$.

Also, let $\quad b^{\frac{1}{q}}=y$, whence $b^{\frac{p}{q}}=y^p$, and $b=y^q$.

Then, substituting, we have

$$\frac{a^n-b^n}{a-b}=\frac{a^{\frac{p}{q}}-b^{\frac{p}{q}}}{a-b}=\frac{x^p-y^p}{x^q-y^q}=\frac{\dfrac{x^p-y^p}{x-y}}{\dfrac{x^q-y^q}{x-y}}.$$

But p and q are positive integers; therefore, when $a=b$, and, consequently, $x=y$, according to case first, the numerator of the last fraction becomes px^{p-1}, and the denominator becomes qx^{q-1}; that is, the fraction reduces to

$$\frac{px^{p-1}}{qx^{q-1}}, \text{ or } \frac{p}{q}x^{p-q}.$$

Substituting for x its value $a^{\frac{1}{q}}$, the fraction reduces to

$$\frac{p}{q}a^{\frac{p-q}{q}}, \text{ or } \frac{p}{q}a^{\frac{p}{q}-1}, \text{ or } na^{n-1}.$$

Third. Suppose n to be *negative*, and either integral or fractional; or let $n=-m$. Then we shall have

$$\frac{a^n-b^n}{a-b}=\frac{a^{-m}-b^{-m}}{a-b}=\frac{\frac{1}{a^m}-\frac{1}{b^m}}{a-b}=\frac{1}{a^mb^m}\cdot\frac{b^m-a^m}{a-b}=-\frac{1}{a^mb^m}\cdot\frac{a^m-b^m}{a-b}.$$

Now, when $a=b$, the first factor of the last expression reduces to $-\frac{1}{a^{2m}}$, or $-a^{-2m}$, and the second factor (by one of the preceding cases) reduces to ma^{m-1}. Hence the expression becomes $-a^{-2m}\times ma^{m-1}$, or $-ma^{-m-1}$, or na^{n-1}.

387. It is required to obtain *a general formula expressing the value of* $(x+a)^m$, whether m be positive or negative, integral or fractional.

Now $x+a=x\left(1+\frac{a}{x}\right)$; therefore $(x+a)^m=x^m\left(1+\frac{a}{x}\right)^m$.

If then we obtain the development of $\left(1+\frac{a}{x}\right)^m$, we have only to multiply it by x^m to obtain that of $(x+a)^m$.

Let $\frac{a}{x}=z$; then, to develop $(1+z)^m$, assume

$$(1+z)^m=A+Bz+Cz^2+Dz^3+, \text{ etc.,} \qquad (1.)$$

in which A, B, C, D, etc., are coefficients independent of z, and we are to determine their values.

Now this equation must be true for any value of z; it must therefore be true when $z=0$, in which case $A=1$.

Substituting this value of A in Eq. (1), it becomes

$$(1+z)^m=1+Bz+Cz^2+Dz^3+, \text{ etc.} \qquad (2.)$$

Since Eq. (2) is to be true for all values of z, let $z=n$; then (2) becomes

$$(1+n)^m=1+Bn+Cn^2+Dn^3+, \text{ etc.} \qquad (3.)$$

Subtracting (3) from (2), member from member, we have

$$(1+z)^m-(1+n)^m=B(z-n)+C(z^2-n^2)+D(z^3-n^3)+, \text{ etc.} \qquad (4.)$$

Dividing the first member of (4) by $(1+z)-(1+n)$, and the second by its equal $z-n$, we have

$$\frac{(1+z)^m-(1+n)^m}{(1+z)-(1+n)}=B+C\frac{z^2-n^2}{z-n}+D\frac{z^3-n^3}{z-n}+, \text{ etc.} \quad (5.)$$

But when $z=n$, or $1+z=1+n$, the first member of equation (5) becomes $m(1+z)^{m-1}$.

Also, $\frac{z^2-n^2}{z-n}=z+n$, when $z=n$, becomes $2z$.

$\frac{z^3-n^3}{z-n}=z^2+zn+n^2$, when $z=n$, becomes $3z^2$, etc.

These values substituted in (5) give

$$m(1+z)^{m-1}=B+2Cz+3Dz^2+4Ez^3+, \text{ etc.} \quad (6.)$$

Multiplying both members of Eq. (6) by $1+z$, we have

$$m(1+z)^m=B+(2C+B)z+(3D+2C)z^2+(4E+3D)z^3+, \text{etc.} \quad (7.)$$

If we multiply Eq. (2) by m, we have

$$m(1+z)^m=m+mBz+mCz^2+mDz^3+, \text{ etc.} \quad (8.)$$

The first members of Eq. (7) and (8) are equal; hence their second members are also equal, and we have

$$m+mBz+mCz^2+mDz^3+, \text{ etc.}=$$
$$B+(2C+B)z+(3D+2C)z^2+(4E+3D)z^3+, \text{ etc.} \quad (9.)$$

This equation is an identical equation; that is, it is true for all values of z. Therefore the coefficients of the like powers of z in the two members are equal each to each, and we have

$$B=m.$$

$$2C+B=mB, \text{ whence } C=\frac{m(m-1)}{2};$$

$$3D+2C=mC, \quad \text{"} \quad D=\frac{m(m-1)(m-2)}{2.3}, \text{ etc.}$$

Substituting these values in (2), we have

$$(1+z)^m=1+mz+\frac{m(m-1)}{2}z^2+\frac{m(m-1)(m-2)}{2.3}z^3+, \text{ etc.} \quad (10.)$$

If in this equation we restore the value of z, which is $\frac{a}{x}$, we have

$$\left(1+\frac{a}{x}\right)^m=1+m.\frac{a}{x}+\frac{m(m-1)}{2}.\frac{a^2}{x^2}+\frac{m(m-1)(m-2)}{2.3}.\frac{a^3}{x^3}+, \text{ etc.};$$

and multiplying both members by x^m, we obtain

$$(x+a)^m = x^m + mx^{m-1}a + \frac{m(m-1)}{2}x^{m-2}a^2 + \frac{m(m-1)(m-2)}{2.3}x^{m-3}a^3 +, \text{ etc.,} \qquad (11.)$$

which is the general formula for the development of any binomial $(x+a)^m$, whatever be the values of x and a, and whether m be positive or negative, integral or fractional; and this formula is known as the *Binomial Theorem of Sir Isaac Newton.*

388. *When the Series is Finite.*—The preceding development is a series of an infinite number of terms; but when m is a positive integer, the series will terminate at the $(m+1)$th term, and all the succeeding terms will become zero. For the second term of Eq. (11) contains the factor m, the third term the factor $m-1$, the fourth term the factor $m-2$, and the $(m+2)$d term contains the factor $m-m$, or 0, which reduces that term to 0; and since all the succeeding terms also contain the same factor, they also become 0. There will therefore remain only $m+1$ terms.

When m is not a positive integer, it is evident that no one of the factors m, $m-1$, $m-2$, $m-3$, etc., can be equal to 0, so that in that case the development will be an infinite series.

389. *Expansion of Binomials with negative integral Exponents.* This is effected by substitution in formula (11).

Ex. 1. Expand $\frac{1}{a+b}$ or $(a+b)^{-1}$ into an infinite series.

In (11) let $m=-1$, and we find

the coefficient of the second term is -1,

" " third " is $\frac{-1\times-2}{2}=+1$,

" " fourth " is $\frac{+1\times-3}{3}=-1$,

" " fifth " is $\frac{-1\times-4}{4}=+1$, etc.

Hence we have

$$(a+b)^{-1}=a^{-1}-a^{-2}b+a^{-3}b^2-a^{-4}b^3+, \text{ etc.},$$

or

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+, \text{ etc.},$$

which is an infinite series, and the law of the series is obvious. We might have obtained the same result by the ordinary method of division.

Ex. 2. Expand $\frac{1}{(a+b)^2}$ or $(a+b)^{-2}$ into an infinite series.

Ans. $a^{-2}-2a^{-3}b+3a^{-4}b^2-4a^{-5}b^3+5a^{-6}b^4-$, etc.,

or

$$\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-, \text{ etc.},$$

where the law of the series is obvious.

Ex. 3. Expand $\frac{1}{a-b}$ or $(a-b)^{-1}$ into an infinite series.

Ans. $a^{-1}+a^{-2}b+a^{-3}b^2+a^{-4}b^3+$, etc.

Ex. 4. Expand $\frac{1}{(a-b)^2}$ or $(a-b)^{-2}$ into an infinite series.

Ex. 5. Expand $(a+b)^{-3}$ into an infinite series.

Ans. $a^{-3}-3a^{-4}b+6a^{-5}b^2-10a^{-6}b^3+15a^{-7}b^4-$, etc.

Ex. 6. Expand $(a-b)^{-4}$ into an infinite series.

Ans. $a^{-4}+4a^{-5}b+10a^{-6}b^2+20a^{-7}b^3+35a^{-8}b^4+$, etc.

Ex. 7. Expand $(1+2x)^{-5}$ into an infinite series.

Ans. $1-10x+60x^2-280x^3+$, etc.

390. *Expansion of Binomials with positive Fractional Fxponents.*

Ex. 1. Expand $\sqrt{a+b}$ or $(a+b)^{\frac{1}{2}}$ into an infinite series.

Represent the coefficients of the different terms by A, B, C, D, etc.; then

$$A= \quad +1,$$
$$B= \quad n \quad =+\tfrac{1}{2},$$
$$C=B\times\frac{n-1}{2}=-\frac{1}{2.4},$$
$$D=C\times\frac{n-2}{3}=+\frac{1.3}{2.4.6},$$
$$E=D\times\frac{n-3}{4}=-\frac{1.3.5}{2.4.6.8}, \text{ etc.}$$

Hence we have

$$(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}b-\frac{1}{2.4}a^{-\frac{3}{2}}b^2+\frac{1.3}{2.4.6}a^{-\frac{5}{2}}b^3-\frac{1.3.5}{2.4.6.8}a^{-\frac{7}{2}}b^4 +, \text{ etc.}$$

The factors which form the coefficients are kept distinct, in order to show more clearly the *law* of the series. The numerators of the coefficients contain the series of odd numbers, 1, 3, 5, 7, etc., while the denominators contain the even numbers, 2, 4, 6, 8, etc.

Ex. 2. Expand $(x-a)^{\frac{1}{2}}$ into an infinite series.

Ex. 3. Expand $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

$$\textit{Ans. } a+\frac{a^{-1}x}{2}-\frac{a^{-3}x^2}{2.4}+\frac{3a^{-5}x^3}{2.4.6}-\frac{3.5a^{-7}x^4}{2.4.6.8}+, \text{ etc.}$$

or

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+, \text{ etc.}$$

Ex. 4. Expand $(a+b)^{\frac{1}{3}}$ into an infinite series.

$$\textit{Ans. } a^{\frac{1}{3}}\left\{1+\frac{b}{3a}-\frac{2b^2}{3.6a^2}+\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}+, \text{ etc.}\right\}$$

Ex. 5. Expand $(a^3-b^3)^{\frac{1}{3}}$ into an infinite series.

$$\textit{Ans. } a\left\{1-\frac{b^3}{3a^3}-\frac{2b^6}{3.6a^6}-\frac{2.5b^9}{3.6.9a^9}-, \text{ etc.}\right\}$$

Ex. 6. Expand $(a+x)^{\frac{2}{3}}$ into an infinite series.

Ex. 7. Expand $(a-b)^{\frac{1}{4}}$ into an infinite series.

$$\textit{Ans. } a^{\frac{1}{4}}\left\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-, \text{ etc.}\right\}$$

Ex. 8. Expand $(1-x)^{\frac{1}{5}}$ into an infinite series.

$$\textit{Ans. } 1-\frac{x}{5}-\frac{4x^2}{5.10}-\frac{4.9x^3}{5.10.15}-\frac{4.9.14x^4}{5.10.15.20}-, \text{ etc.}$$

391. *Expansion of Binomials with negative Fractional Exponents.*

Ex. 1. Expand $\frac{1}{(a+b)^{\frac{1}{2}}}$ or $(a+b)^{-\frac{1}{2}}$ into an infinite series.

The terms without the coefficients are

$$a^{-\frac{1}{2}},\ a^{-\frac{3}{2}}b,\ a^{-\frac{5}{2}}b^2,\ a^{-\frac{7}{2}}b^3,\ a^{-\frac{9}{2}}b^4, \text{ etc.}$$

Represent the coefficients by A, B, C, D, etc.; then

$$\begin{aligned}
A &= \qquad\qquad +1,\\
B &= \quad n \quad = -\tfrac{1}{2},\\
C &= B\times\frac{n-1}{2} = +\frac{1.3}{2.4},\\
D &= C\times\frac{n-2}{3} = -\frac{1.3.5}{2.4.6},\\
E &= D\times\frac{n-3}{4} = +\frac{1.3.5.7}{2.4.6.8}, \text{ etc.}
\end{aligned}$$

Hence we obtain

$$(a+b)^{-\frac{1}{2}} = a^{-\frac{1}{2}} - \frac{1}{2}a^{-\frac{3}{2}}b + \frac{1.3}{2.4}a^{-\frac{5}{2}}b^2 - \frac{1.3.5}{2.4.6}a^{-\frac{7}{2}}b^3 + \frac{1.3.5.7}{2.4.6.8}a^{-\frac{9}{2}}b^4$$

$$-, \text{ etc.}$$

$$= \frac{1}{\sqrt{a}}\left\{1 - \frac{b}{2a} + \frac{3b^2}{2.4a^2} - \frac{3.5b^3}{2.4.6a^3} + \frac{3.5.7b^4}{2.4.6.8a^4} -, \text{ etc.}\right\}$$

Ex. 2. Expand $(a^2-x)^{-\frac{1}{2}}$ into an infinite series.

$$\textit{Ans. } \frac{1}{a} + \frac{x}{2a^3} + \frac{1.3x^2}{2.4a^5} + \frac{1.3.5x^3}{2.4.6a^7} + \frac{1.3.5.7x^4}{2.4.6.8a^9} +, \text{ etc.}$$

Ex. 3. Expand $\frac{m}{\sqrt{x^2+a^4}}$ into an infinite series.

$$\textit{Ans. } \frac{m}{x}\left\{1 - \frac{a^4}{2x^2} + \frac{1.3a^8}{2.4x^4} - \frac{1.3.5a^{12}}{2.4.6x^6} + \frac{1.3.5.7a^{16}}{2.4.6.8x^8} -, \text{ etc.}\right\}$$

Ex. 4. Expand $(a+x)^{-\frac{1}{3}}$ into an infinite series.

$$\textit{Ans. } a^{-\frac{1}{3}} - \frac{1}{3}a^{-\frac{4}{3}}x + \frac{1.4}{3.6}a^{-\frac{7}{3}}x^2 - \frac{1.4.7}{3.6.9}a^{-\frac{10}{3}}x^3 + \frac{1.4.7.10}{3.6.9.12}a^{-\frac{13}{3}}x^4 -, \text{ etc.}$$

Ex. 5. Expand $(a^2-x^2)^{-\frac{1}{4}}$ into an infinite series.

$$\textit{Ans. } \frac{1}{\sqrt{a}}\left\{1 + \frac{x^2}{4a^2} + \frac{1.5x^4}{4.8a^4} + \frac{1.5.9x^6}{4.8.12a^6} +, \text{ etc.}\right\}$$

Ex. 6. Expand $(1+x)^{-\frac{1}{5}}$ into an infinite series.

$$\textit{Ans. } 1 - \frac{x}{5} + \frac{6x^2}{5.10} - \frac{6.11x^3}{5.10.15} + \frac{6.11.16x^4}{5.10.15.20} -, \text{ etc.}$$

392. *Extraction of any Root of a Surd Number.*—The approximate value of a surd root may be found by the binomial theorem by dividing the number into two parts, and considering it as a binomial.

Ex. 1. Find the square root of 10.

$$\sqrt{10}=\sqrt{9+1}=(9+1)^{\frac{1}{2}}.$$

If, in Ex. 3, Art. 390, we make $a^2=9$ and $x=1$, we shall have

$$\sqrt{9+1}=3+\frac{1}{2.3}-\frac{1}{2.4.3^3}+\frac{3}{2.4.6.3^5}-\frac{3.5}{2.4.6.8.3^7}+\frac{3.5.7}{2.4.6.8.10.3^9}-\text{, etc.}$$

The value of the	first	term	is		3.0000000
"	" second	"		+	.1666667
"	" third	"		−	.0046296
"	" fourth	"		+	.0002572
"	" fifth	"		−	.0000179
"	" sixth	"		+	.0000014
"	" seventh	"		−	.0000001
Their sum is					3.1622777,

which is the square root of 10 correct to seven decimal places.

Ex. 2. Find the square root of 99.

$$\sqrt{99}=\sqrt{100-1}=(100-1)^{\frac{1}{2}}.$$

Substituting in Ex. 3, Art. 390, we have

$$\sqrt{99}=10-\frac{1}{2.10}-\frac{1}{2.4.10^3}-\frac{3}{2.4.6.10^5}-\text{, etc.}$$

The value of the	first	term	is		10.0000000
"	" second	"		−	.0500000
"	" third	"		−	.0001250
"	" fourth	"		−	.0000006
Their sum is					9.9498744,

which is the square root of 99 correct to seven decimal places.

393. The method here exemplified for finding the nth root of any number is expressed in the following

RULE.

Find, by trial, the nearest integral root (a), and divide the given number into two parts, one of which is the nth power of (a). Con-

sider these two parts as the terms of a binomial, and develop it into a series by the binomial theorem.

Ex. 3. Find the cube root of 9 to seven decimal places.

$$\textit{Ans. } 2+\frac{1}{3.2^2}-\frac{2}{3.6.2^5}+\frac{2.5}{3.6.9.2^8}-\frac{2.5.8}{3.6.9.12.2^{11}}+\text{, etc.,}$$

$$=2.0800838.$$

Ex. 4. Find the cube root of 31 to seven decimal places.

$$\textit{Ans. } 3\left\{1+\frac{4}{3.27}-\frac{2.4^2}{3.6.27^2}+\frac{2.5.4^3}{3.6.9.27^3}-\frac{2.5.8.4^4}{3.6.9.12.27^4}+\text{, etc.,}\right\}.$$

$$=3.1413806.$$

Ex. 5. Find the fifth root of 30 to seven decimal places.

$$\textit{Ans. } 2-\frac{2}{5.16}-\frac{2.4}{5.10.16^2}-\frac{2.4.9}{5.10.15.16^3}-\text{, etc.,}$$

$$=1.9743506.$$

CHAPTER XX.

LOGARITHMS.

394. The *logarithm* of a number is the exponent of the power to which a constant number must be raised in order to be equal to the proposed number. The constant number is called the *base of the system.*

Thus, if a denote any positive number except unity, and $a^2 = m$, then 2 is the exponent of the power to which a must be raised to equal m; that is, 2 is the logarithm of m in the system whose base is a. If $a^x = m$, then x is the logarithm of m in the system whose base is a.

395. If we suppose a to remain constant while m assumes in succession every value from zero to infinity, the corresponding values of x will constitute a *system of logarithms.*

Since an indefinite number of different values may be attributed to a, it follows that *there may be an indefinite number of systems of logarithms.* Only two systems, however, have come into general use, viz., that system whose base is 10, called Briggs's system, or the *common* system of logarithms; and that system whose base is 2.718+, called the Naperian system, or *hyperbolic* system of logarithms.

Properties of Logarithms in general.

396. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.*

Let a denote the base of the system; also, let m and n be any two numbers, and x and y their logarithms. Then, by the definition of logarithms, we have

$$a^x = m, \qquad (1.)$$

$$a^y = n. \qquad (2.)$$

Multiplying together equations (1) and (2) member by member, we have $a^{x+y} = mn.$

Therefore, according to the definition of logarithms, $x+y$ is the logarithm of mn, since it is the exponent of that power of the base which is equal to mn.

For convenience, we will use log. to denote logarithm, and we have

$$x+y=\log.\ mn=\log.\ m+\log.\ n.$$

Hence we see that if it is required to multiply two or more numbers together, we have only to take their logarithms from a table and *add* them together; then find the number corresponding to the resulting logarithm, and it will be the product required.

397. *The logarithm of the quotient of two numbers is equal to the logarithm of the dividend diminished by that of the divisor.*

If we divide Eq. (1) by Eq. (2), member by member, we shall have

$$a^{x-y}=\frac{m}{n}.$$

Therefore, according to the definition, $x-y$ is the logarithm of $\frac{m}{n}$, since it is the exponent of that power of the base a which is equal to $\frac{m}{n}$. That is,

$$x-y=\log.\left(\frac{m}{n}\right)=\log.\ m-\log.\ n.$$

Hence we see that if we wish to divide one number by another, we have only to take their logarithms from the table and subtract the logarithm of the divisor from that of the dividend; then find the number corresponding to the resulting logarithm, and it will be the quotient required.

398. *The logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power.*

If we raise both members of Eq. (1) to any power denoted by p, we have $a^{px}=m^p$.

Therefore, according to the definition, px is the logarithm of m^p, since it is the exponent of that power of the base which is equal to m^p. That is,

$$px=\log.\ (m^p)=p\ \log.\ m.$$

Therefore, to involve a given number to any power, we multiply the logarithm of the number by the exponent of the power; the product is the logarithm of the required power.

399. *The logarithm of any root of a number is equal to the logarithm of that number divided by the index of the root.*

If we extract the rth root of both members of Eq. (1), we shall have

$$a^{\frac{x}{r}}=\sqrt[r]{m}.$$

Therefore, according to the definition, $\frac{x}{r}$ is the logarithm of $\sqrt[r]{m}$. That is

$$\frac{x}{r}=\text{log. }\sqrt[r]{m}=\frac{\text{log. } m}{r}.$$

Therefore, to extract any root of a number, we divide the logarithm of the number by the index of the root; the quotient is the logarithm of the required root.

400. The following examples will show the application of the preceding principles:

Ex. 1. $\text{log. }(abcd)=\text{log. } a+\text{log. } b+\text{log. } c+\text{log. } d.$

Ex. 2. $\text{log. }\left(\frac{abc}{de}\right)=\text{log. } a+\text{log. } b+\text{log. } c-\text{log. } d-\text{log. } e.$

Ex. 3. $\text{log. }(a^m b^n c^p)=m\text{ log. } a+n\text{ log. } b+p\text{ log. } c.$

Ex. 4. $\text{log. }\left(\frac{a^m b^n}{c^p}\right)=m\text{ log. } a+n\text{ log. } b-p\text{ log. } c.$

Ex. 5. $\text{log. }\sqrt{ab}=\frac{1}{2}(\text{log. } a+\text{log. } b).$

Ex. 6. $\text{log. }\sqrt[3]{\frac{ab^2c^4}{d^5}}=\frac{1}{3}\{\text{log. } a+2\text{ log. } b+4\text{ log. } c-5\text{ log. } d\}.$

Ex. 7. $\text{log. }(a^3\sqrt[4]{a^3})=\text{log. }\left(a^{\frac{15}{4}}\right)=\frac{15}{4}\text{ log. } a.$

Ex. 8. $\text{log.}(a^2-x^2)=\text{log.}\{(a+x)(a-x)\}=\text{log. }(a+x)+\text{log.}(a-x).$

Ex. 9. $\text{log. }\sqrt{a^2-x^2}=\frac{1}{2}\text{ log. }(a+x)+\frac{1}{2}\text{ log. }(a-x).$

Ex. 10. $\text{log. }\left(\frac{\sqrt{3}\times\sqrt[3]{4}}{\sqrt[5]{6}\times\sqrt[7]{2}}\right)=\frac{1}{2}\text{ log. } 3+\frac{1}{3}\text{ log. } 4-\frac{1}{5}\text{ log. } 6-\frac{1}{7}\text{ log. } 2.$

401. *In all systems of logarithms, the logarithm of unity is zero.*

For in the equation $a^x = n$,
if we make $n=1$, the corresponding value of x will be 0, since $a^0=1$, Art. 75; that is, log. $1=0$.

402. *In all systems of logarithms, the logarithm of the base is unity.*

For $a^1=a$;
that is, log. $a=1$.

Common Logarithms.

403. Since the base of the common system of logarithms is 10, all numbers in this system are to be regarded as *powers of* 10. Thus, since

$$10^0=1, \quad \text{we have} \quad \text{log. } 1=0;$$
$$10^1=10, \quad \text{“} \quad \text{log. } 10=1;$$
$$10^2=100, \quad \text{“} \quad \text{log. } 100=2;$$
$$10^3=1000, \quad \text{“} \quad \text{log. } 1000=3, \text{ etc.}$$

From this it appears that in Briggs's system the logarithm of any number between 1 and 10 is some number between 0 and 1; that is, it is a fraction less than unity, and is generally expressed as a decimal. The logarithm of any number between 10 and 100 is some number between 1 and 2; that is, it is equal to 1 *plus* a decimal. The logarithm of any number between 100 and 1000 is some number between 2 and 3; that is, it is equal to 2 *plus* a decimal; and so on.

404. The same principle may be extended to fractions by means of negative exponents. Thus, since

$$10^{-1}=\tfrac{1}{10} \text{ or } 0.1, \quad \text{we have} \quad \text{log. } 0.1=-1;$$
$$10^{-2}=\tfrac{1}{100} \text{ or } 0.01, \quad \text{“} \quad \text{log. } 0.01=-2;$$
$$10^{-3}=\tfrac{1}{1000} \text{ or } 0.001, \quad \text{“} \quad \text{log. } 0.001=-3;$$
$$10^{-4}=\tfrac{1}{10000} \text{ or } 0.0001, \quad \text{“} \quad \text{log. } 0.0001=-4, \text{ etc.}$$

Hence it appears that the logarithm of every number between 1 and 0.1 is some number between 0 and -1, or may be represented by -1 *plus* a decimal. The logarithm of every number between 0.1 and 0.01 is some number between -1 and

-2, or may be represented by -2 *plus* a decimal. The logarithm of every number between 0.01 and 0.001 is some number between -2 and -3, or may be represented by -3 *plus* a decimal, and so on.

405. Hence we see that the logarithms of most numbers must consist of two parts, an integral part and a decimal part. The former part is called the *characteristic* or *index* of the logarithm. The characteristic may always be determined by the following

RULE.

The characteristic of the logarithm of any number is equal to the number of places by which the first significant figure of that number is removed from the unit's place, and is positive when this figure is to the left, negative when it is to the right, and zero when it is in the unit's place.

Thus the characteristic of the logarithm of 397 is $+2$, and that of 5673 is $+3$, while the characteristic of the logarithm of 0.0046 is -3.

406. *The same decimal part is common to the logarithms of all numbers composed of the same significant figures.*

For, since the logarithm of 10 is 1, it follows from Art. 397 that if a number be divided by 10, its logarithm will be diminished by 1, the decimal part remaining unchanged. Thus, if we denote the decimal part of the logarithm of 3456 by m, we shall have

$\log. 3456 = 3+m.$	$\log. .3456 = -1+m.$
$\log. 345.6 = 2+m.$	$\log. .03456 = -2+m.$
$\log. 34.56 = 1+m.$	$\log. .003456 = -3+m.$
$\log. 3.456 = 0+m.$	$\log. .0003456 = -4+m.$

Table of Logarithms.

407. The table on pages 290, 291, contains the decimal part of the common logarithm of the series of natural numbers from 100 to 999, carried to four decimal places. Since these numbers are all decimals, the decimal point is omitted, and the characteristic is to be supplied according to the rule in Art. 405.

408. *To find the logarithm of any number consisting of not more than three figures.*—Look on one of the pages of the table, along the left-hand column marked No., for the two left-hand figures, and the third figure at the head of one of the other columns. Opposite to the first two figures, and in the column under the third figure, will be found the decimal part of its logarithm. To this must be prefixed the characteristic, according to the rule in Art. 405. Thus

the logarithm of 347 is 2.5403;
" 871 is 2.9400.

The logarithm of 63, or 63.0, is 1.7993;
" 5, or 5.00, is 0.6990;
" 0.235 is $\bar{1}.3711$.

The minus sign is here placed *over* the characteristic, to show that that alone is negative, while the decimal part of the logarithm is positive.

409. *To find the logarithm of any number containing more than three figures.*—By inspecting the table, we shall find that within certain limits the differences of logarithms are proportional to the differences of their corresponding numbers. Thus

the logarithm of 216 is 2.3345;
" 217 is 2.3365;
" 218 is 2.3385.

Here the difference between the successive logarithms, called the *tabular difference*, is constantly 20, corresponding to a difference of unity in the natural numbers. If, then, we suppose the logarithms to increase at the same rate as their corresponding numbers (as they do nearly), a difference of 0.1 in the numbers should correspond to a difference of 2 in the logarithms; a difference of 0.2 in the numbers should correspond to a difference of 4 in the logarithms, etc. Hence

the logarithm of 216.1 must be 2.3347;
" 216.2 " 2.3349, etc.

In order to facilitate the computation, there is given, on the right margin of each page, the proportional part for the fourth figure of the natural number, corresponding to tabular differ-

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

PROPORTIONAL PARTS.

	43	42	41	40	39
1	4	4	4	4	4
2	9	8	8	8	8
3	13	13	12	12	12
4	17	17	16	16	16
5	22	21	21	20	20
6	26	25	25	24	23
7	30	29	29	28	27
8	34	34	33	32	31
9	39	38	37	36	35

	38	37	36	35	34
1	4	4	4	4	3
2	8	7	7	7	7
3	11	11	11	11	10
4	15	15	14	14	14
5	19	19	18	18	17
6	23	22	22	21	20
7	27	26	25	25	24
8	30	30	29	28	27
9	34	33	32	32	31

	33	32	31	30	29
1	3	3	3	3	3
2	7	6	6	6	6
3	10	10	9	9	9
4	13	13	12	12	12
5	17	16	16	15	15
6	20	19	19	18	17
7	23	22	22	21	20
8	26	26	25	24	23
9	30	29	28	27	26

	28	27	26	25	24
1	3	3	3	3	2
2	6	5	5	5	5
3	8	8	8	8	7
4	11	11	10	10	10
5	14	14	13	13	12
6	17	16	16	15	14
7	20	19	18	18	17
8	22	22	21	20	19
9	25	24	23	23	22

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

PROPORTIONAL PARTS.

	23	22	21	20	19
1	2	2	2	2	2
2	5	4	4	4	4
3	7	7	6	6	6
4	9	9	8	8	8
5	12	11	11	10	10
6	14	13	13	12	11
7	16	15	15	14	13
8	18	18	17	16	15
9	21	20	19	18	17

	18	17	16	15	14
1	2	2	2	2	1
2	4	3	3	3	3
3	5	5	5	5	4
4	7	7	6	6	6
5	9	9	8	8	7
6	11	10	10	9	8
7	13	12	11	11	10
8	14	14	13	12	11
9	16	15	14	14	13

	13	12	11	10	9
1	1	1	1	1	1
2	3	2	2	2	2
3	4	4	3	3	3
4	5	5	4	4	4
5	7	6	6	5	5
6	8	7	7	6	5
7	9	8	8	7	6
8	10	10	9	8	7
9	12	11	10	9	8

	8	7	6	5	4
1	1	1	1	1	0
2	2	1	1	1	1
3	2	2	2	2	1
4	3	3	2	2	2
5	4	4	3	3	2
6	5	4	4	3	2
7	6	5	4	4	3
8	6	6	5	4	3
9	7	6	5	5	4

ences from 43 to 4. Thus, on page 291, near the top, we see that when the tabular difference is 20, the corrections for .1, .2, .3, etc., are 2, 4, 6, etc.

It is obvious that the correction for a figure in the fifth place of the natural number must be one tenth of the correction for the same figure if it stood in the fourth place. Such a correction would, however, generally be inappreciable in logarithms which extend only to four decimal places.

EXAMPLES.

Find the logarithm of		4576.	*Ans.* 3.6605.
"	"	13.78.	*Ans.* 1.1392.
"	"	1.682.	*Ans.* 0.2258.
"	"	.03211.	*Ans.* $\overline{2}$.5066.
"	"	.4735.	*Ans.* $\overline{1}$.6753.
"	"	15983.	*Ans.* 4.2036.

The logarithms here given are only approximate. We can obtain the exact logarithm of very few numbers; but by taking a sufficient number of decimals we can approach as nearly as we please to the true logarithm.

410. *To find the natural number corresponding to any logarithm.*—Look in the table for the decimal part of the logarithm, neglecting the characteristic; and if the decimal is exactly found, the first two figures of the corresponding natural number will be found opposite to it in the column headed No., and the third figure will be found at the top of the page. This number must be made to correspond with the characteristic by pointing off decimals or annexing ciphers. Thus

the natural number belonging to the logarithm 3.3692 is 2340;
" " " " 1.5378 is 34.5.

If the decimal part of the logarithm is not exactly contained in the table, look for the *nearest less* logarithm, and take out the three figures of the corresponding natural number as before. The additional figure or figures may be obtained by means of the proportional parts on the margin of the page.

Find the number corresponding to the logarithm 3.3685.

The next less logarithm in the table is .3674, and the three corresponding figures of the natural number are 233. Their logarithm is less than the one proposed by 11, and the tabular difference is 18. By referring to the margin of page 291, we find that, with a difference of 18, the figure corresponding to the proportional part 11 is 6. Hence, since the characteristic of the proposed logarithm is 3, the required natural number is 2336.

EXAMPLES.

1. Find the number corresponding to the logarithm 2.5386.
Ans. 345.6.

2. Find the number corresponding to the logarithm 0.2345.
Ans. 1.716.

3. Find the number corresponding to the logarithm 1.9946.
Ans. 98.76.

4. Find the number corresponding to the logarithm $\bar{1}.6478$.
Ans. 0.4444.

411. *Multiplication by Logarithms.*—According to Art. 396, to find the product of two numbers we have the following

RULE.

Add the logarithms of the factors; the sum will be the logarithm of the product.

The word *sum* is here to be understood in its algebraic sense. The decimal part of a logarithm is invariably positive; but the characteristic may be either positive or negative.

Ex. 1. Find the product of 57.98 by 3.12.

The logarithm of	57.98	is	1.7633.
"	3.12	is	0.4942.
The log. of the product	180.9	is	2.2575.

Ex. 2. Find the product of 0.00563 by 172.5.

The logarithm of	0.00563	is	$\bar{3}.7505$.
"	172.5	is	2.2368.
The log. of the product	0.971	is	$\bar{1}.9873$.

Ex. 3. Find the product of 54.32 by 6.543.

Ex. 4. Find the product of 3.854 by 0.5761.

412. *Division by Logarithms.*—According to Art. 397, to find the quotient of two numbers we have the following

RULE.

From the logarithm of the dividend subtract the logarithm of the divisor; the difference will be the logarithm of the quotient.

The word *difference* is here to be understood in its algebraic sense; the decimal part of the logarithm being invariably positive, while the characteristic may be either positive or negative.

Ex. 1. Find the quotient of 888.7 divided by 42.24.

The logarithm of 888.7 is	2.9488.
" 42.24 is	1.6257.
The quotient is 21.04, whose log. is	1.3231.

Ex. 2. Find the quotient of 0.8692 divided by 42.32.

The logarithm of 0.8692 is	$\bar{1}$.9391.
" 42.32 is	1.6265.
The quotient is 0.002054, whose log. is	$\bar{3}$.3126.

Ex. 3. Find the quotient of 380.7 divided by 13.75.

Ex. 4. Find the quotient of 24.93 divided by .0785.

413. *Involution by Logarithms.*—According to Art. 398, to involve a number to any power we have the following

RULE.

Multiply the logarithm of the number by the exponent of the power required.

It should be remembered that what is carried from the decimal part of the logarithm is positive, whether the characteristic be positive or negative.

Ex. 1. Find the fifth power of 2.846.

The logarithm of 2.846 is	0.4542.
	5
The fifth power is 186.65, whose log. is	2.2710.

Ex. 2. Find the cube of .07654.

The logarithm of .07654 is $\overline{2}.8839$.

$$\begin{array}{r} \overline{2}.8839 \\ 3 \\ \hline \overline{4}.6517 \end{array}$$

The cube is 0.0004484, whose log. is $\overline{4}.6517$.

Ex. 3. Find the 20th power of 1.06.

Ex. 4. Find the seventh power of 0.8952.

414. *Evolution by Logarithms.*—According to Art. 399, to extract any root of a number we have the following

RULE.

Divide the logarithm of the number by the index of the root required.

Ex. 1. Find the cube root of 482.4.

The logarithm of 482.4 is 2.6834.

Dividing by 3, we have 0.8945, which corresponds to 7.843, which is therefore the root required.

Ex. 2. Find the 100th root of 365. *Ans.* 1.061.

When the characteristic of the logarithm is negative, and is not divisible by the given divisor, we may increase the characteristic by any number which will make it exactly divisible, provided we prefix an equal positive number to the decimal part of the logarithm.

Ex. 3. Find the seventh root of 0.005846.

The logarithm of 0.005846 is $\overline{3}.7669$, which may be written

$$\overline{7}+4.7669.$$

Dividing by 7, we have $\overline{1}.6810$, which is the logarithm of .4797, which is therefore the root required.

Ex. 4. Find the 10th root of 0.007815.

415. *Proportion by Logarithms.*—The fourth term of a proportion is found by multiplying together the second and third terms and dividing by the first. Hence, to find the fourth term of a proportion by logarithms, we have the following

RULE.

Add the logarithms of the second and third terms, and from their sum subtract the logarithm of the first term.

Ex. 1. Find a fourth proportional to 72.34, 2.519, and 357.5.
Ans. 12.45.

Ex. 2. Find a fourth proportional to 43.17, 275, and 5.762.

Ex. 3. Find a fourth proportional to 5.745, 781.2, and 54.27.

Exponential Equations.

416. *An exponential equation is one in which the unknown quantity occurs as an exponent.* Thus,

$$a^x = b$$

is an exponential equation, from which, when a and b are known, the value of x may be found. If $a=2$ and $b=8$, the equation becomes $$2^x = 8,$$
in which the value of x is evidently 3, since $2^3=8$.

If $a=16$ and $b=2$, the equation becomes

$$16^x = 2,$$

in which the value of x is evidently $\frac{1}{4}$, since $16^{\frac{1}{4}}=2$.

417. *Solution by Logarithms.*—When b is not an exact power or root of a, the equation is most readily solved by means of logarithms. Taking the logarithm of each member of the equation $a^x = b$, we have

$$x \log. a = \log. b,$$

whence $$x = \frac{\log. b}{\log. a}.$$

Ex. 1. Solve the equation $3^x = 20$.

$$x = \frac{\log. 20}{\log. 3} = \frac{1.3010}{.4771} = 2.727 \text{ nearly.}$$

Ex. 2. Solve the equation $5^x = 12$.

Ex. 3. Solve the equation $\left(\frac{2}{3}\right)^x = \frac{3}{4}$.

Ex. 4. Solve the equation $10^x = 7$.

Ex. 5. Solve the equation $12^{\frac{1}{x}} = 3$.

Ex. 6. Solve the equation $12^{\frac{3}{x}} = 7$.

Compound Interest.

418. *Interest* is money paid for the use of money. When the interest, as soon as it becomes due, is added to the principal, and interest is charged upon the whole, it is called *compound interest.*

419. *To find the amount of a given sum in any time at compound interest.* It is evident that \$1.00 at 5 per cent. interest becomes at the end of the year a principal of \$1.05; and, since the amount at the end of each year must be proportioned to the principal at the beginning of the year, the amount at the end of two years will be given by the proportion

$$1.00 : 1.05 :: 1.05 : (1.05)^2.$$

The sum (1.05^2) must now be considered as the principal, and the amount at the end of three years will be given by the proportion

$$1.00 : 1.05 :: (1.05)^2 : (1.05)^3.$$

In the same manner, we find that the amount of \$1.00 for n years at 5 per cent. compound interest is $(1.05)^n$.

For the same reason, the amount for n years at 6 per cent. is $(1.06)^n$. It is also evident that the amount of P dollars for a given time must be P times the amount of one dollar.

Hence, if we put

P to represent the principal,
r the interest of one dollar for one year,
n the number of years for which interest is taken,
A the amount of the given principal for n years,

we shall have $$A = P(1+r)^n.$$

This equation contains four quantities, A, P, n, r, any three of which being given, the fourth may be found. The computation is most readily performed by means of logarithms. Taking the logarithms of both members of the preceding equation and reducing, we find

$$\log. A = \log. P + n \times \log. (1+r),$$
$$\log. P = \log. A - n \times \log. (1+r),$$
$$\log. (1+r) = \frac{\log. A - \log. P}{n},$$
$$n = \frac{\log. A - \log. P}{\log. (1+r)}.$$

Ex. 1. How much would 500 dollars amount to in five years at 6 per cent. compound interest?

The log. of 1.06 is	0.0253
	5
	0.1265
The log. of 500 is	2.6990
The amount is \$669.10, whose log. is	2.8255.

Ex. 2. What principal at 6 per cent. compound interest will amount to 500 dollars in seven years? *Ans.* \$332.60.

Ex. 3. At what rate per cent. must 500 dollars be put out at compound interest so that it may amount to \$680.30 in seven years? *Ans.* $4\frac{1}{2}$ per cent.

Ex. 4. In what time will 500 dollars amount to 900 dollars at 6 per cent. compound interest? *Ans.* $10\frac{1}{11}$ years.

Ex. 5. How much would 400 dollars amount to in nine years at 5 per cent. compound interest?

Ex. 6. What principal at 5 per cent. compound interest will amount to 400 dollars in eight years?

Ex. 7. At what rate per cent. must 400 dollars be put out at compound interest so that it may amount to \$620.70 in nine years?

Ex. 8. In what time will a sum of money double at 6 per cent. compound interest?

Ex. 9. In what time will a sum of money double at 5 per cent. compound interest?

Annuities.

420. *An annuity* is a sum of money stipulated to be paid annually, and to continue for a given number of years, for life, or forever.

421. *To find the amount of an annuity left unpaid for any number of years, allowing compound interest.*

Let a denote the annuity, n the number of years, r the interest of one dollar for one year, and A the required amount.

The amount due at the end of the first year is a.

At the end of the second year the amount of the first annuity is $a(1+r)$, and a second payment becomes due; hence the whole sum due at the end of the second year is $a+a(1+r)$.

At the end of the third year a third payment a becomes due, together with the interest on $a+a(1+r)$; hence the whole sum due at the end of the third year is $a+a(1+r)+a(1+r)^2$, or $a\{1+(1+r)+(1+r)^2\}$, and so on.

Hence the amount due at the end of n years is

$$a\{1+(1+r)+(1+r)^2+(1+r)^3+\ldots\ldots +(1+r)^{n-1}\}.$$

These terms form a geometrical progression in which the ratio is $1+r$. Hence, by Art. 332, the sum of the series is

$$A=a.\frac{(1+r)^n-1}{r}.$$

422. *To find the present value of an annuity, to continue for a certain number of years, allowing compound interest.*

The present value of the annuity must be such a sum as, if put out to interest for n years at the rate r, would amount to the same as the amount of the annuity at the end of that period.

If P denote the present value of the annuity, then the amount of the annuity will be $P(1+r)^n$, which must be equal to

$$a.\frac{(1+r)^n-1}{r}.$$

Therefore $$P=\frac{a}{r}.\frac{(1+r)^n-1}{(1+r)^n}.$$

Ex. 1. How much will an annuity of 500 dollars amount to in 15 years at four per cent. compound interest?

$(1+r)^n \quad =1.7987$

$(1+r)^n-1= \ .7987$, whose log. is $\bar{1}.9024$

the log. of .04 is $\bar{2}.6021$

1.3003

the log. of 500 is 2.6990

The amount is \$9983, whose log. is 3.9993.

Ex. 2. What is the present value of an annuity of 500 dollars to continue for 20 years, interest being allowed at the rate of four per cent. per annum?

$$(1+r)^n \quad =2.188$$
$$(1+r)^n-1=1.188, \text{ whose log. is } 0.0748$$
$$\text{the log. of } (1+r)^n \text{ is } \underline{0.3400}$$
$$\bar{1}.7348$$
$$\frac{a}{r}=12500, \text{ whose log. is } 4.0969$$

The present value is \$6787, whose log. is $\overline{3.8317}$.

Ex. 3. How much will an annuity of 600 dollars amount to in 12 years at three per cent. compound interest?

Ex. 4. What is the present value of an annuity of 600 dollars to continue for 12 years at three per cent. compound interest?

Ex. 5. In what time will an annuity of 500 dollars amount to 5000 dollars at 4 per cent. compound interest?

Ans. In $8\frac{3}{5}$ years.

Increase of Population.

423. The natural increase of population in a country is sometimes computed in the same way as compound interest. Knowing the population at two different dates, we compute the *rate of increase* by Art. 419, and from this we may compute the population at any future time on the supposition of a uniform rate of increase. Such computations, however, are not very reliable, for in some countries the population is stationary, and in others it is decreasing.

Ex. 1. The number of the inhabitants of the United States in 1790 was 3,930,000, and in 1860 it was 31,445,000. What was the average increase for every ten years?

Ans. $34\frac{3}{5}$ per cent.

Ex. 2. Suppose the rate of increase to remain the same for the next ten years, what would be the number of inhabitants in 1870? *Ans.* 42,330,000.

Ex. 3. At the same rate, in what time would the number in 1860 be doubled? *Ans.* $23\frac{1}{3}$ years.

Ex. 4. At the same rate, in what time would the number in 1860 be tripled?

To find the Logarithm of any given Number.

424. If m and n denote any two numbers, and x and y their logarithms, then $\frac{x+y}{2}$ will be the logarithm of $\sqrt{mn}$. For, according to Art. 396, $a^{x+y}=mn$, and, taking the square root of each member, we have $a^{\frac{x+y}{2}}=\sqrt{mn}$. Therefore, $\frac{x+y}{2}$ is the logarithm of $\sqrt{mn}$, since it is the exponent of that power of the base which is equal to $\sqrt{mn}$.

Now, in Briggs's system, the logarithm of 10 is 1, of 100 is 2, etc. Hence the logarithm of $\sqrt{10 \times 100}$ is $\frac{1+2}{2}$; that is, the logarithm of 31.6228 is 1.5.

So, also, the logarithm of $\sqrt{10 \times 31.6228}$ is $\frac{1+1.5}{2}$; that is, the logarithm of 17.7828 is 1.25, and so on for any number of logarithms.

In this manner were the first logarithmic tables computed; but more expeditious methods have since been discovered. It is found more convenient to express the logarithm of a number in the form of a *series*.

425. *Logarithms computed by Series.*—The computation of logarithms by series requires the solution of the equation

$$a^x=n,$$

in which a is the base of the system, n any number, and x is the logarithm of that number. In order that a and n may be expanded into a series by the binomial theorem, we will convert them into binomials, and assume $a=1+b$ and $n=1+m$; then we shall have

$$(1+b)^x=1+m,$$

where x is the logarithm of $1+m$, to the base $1+b$, or a.

Involving each member to a power denoted by y, we have

$$(1+b)^{xy}=(1+m)^y.$$

Expanding both members by the binomial theorem, we have

$$1+xyb+\frac{xy(xy-1)}{2}b^2+\frac{xy(xy-1)(xy-2)}{2.3}b^3+, \text{ etc.}=$$

$$1+ym+\frac{y(y-1)}{2}m^2+\frac{y(y-1)(y-2)}{2.3}m^3+, \text{ etc.}$$

Canceling unity from both members and dividing by y, we have

$$x(b+\frac{xy-1}{2}b^2+\frac{(xy-1)(xy-2)}{2.3}b^3+, \text{ etc.})=$$

$$m+\frac{y-1}{2}m^2+\frac{(y-1)(y-2)}{2.3}m^3+, \text{ etc.}$$

This equation is true for all values of y; it will therefore be true when $y=0$. Upon this supposition, the equation becomes

$$x(b-\frac{b^2}{2}+\frac{b^3}{3}-, \text{ etc.})=m-\frac{m^2}{2}+\frac{m^3}{3}-, \text{ etc.},$$

whence $$x=\text{log. }(1+m)=\frac{m-\frac{m^2}{2}+\frac{m^3}{3}-, \text{ etc.}}{b-\frac{b^2}{2}+\frac{b^3}{3}-, \text{ etc.}}.$$

If we put $$\text{M}=\frac{1}{b-\frac{b^2}{2}+\frac{b^3}{3}-, \text{ etc.}},$$

the last equation becomes

$$x=\text{log. }n=\text{log. }(1+m)=\text{M}(m-\frac{m^2}{2}+\frac{m^3}{3}-, \text{ etc.}). \quad (1.)$$

We have thus obtained an expression for the logarithm of the number $1+m$ or n. This expression consists of two factors, viz., the quantity M, which is constant, since it depends simply upon the base of the system; and the quantity within the parenthesis, which depends upon the proposed number. The constant factor M is called the *modulus* of the system.

426. *To determine the Base of Napier's System.*—In Napier's system of logarithms the modulus is assumed equal to *unity*. From this condition the base may be determined. Equation (1), Art. 425, in this case becomes

$$x=m-\frac{m^2}{2}+\frac{m^3}{3}-\frac{m^4}{4}+, \text{ etc.}$$

Reverting this series, Art. 383, Ex. 3, we obtain

$$m = x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +, \text{ etc.}$$

But, by hypothesis, $a^x = n = 1 + m$; therefore

$$a^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{2.3.4} +, \text{ etc.}$$

If x be taken equal to unity, we have

$$a = 2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} +, \text{ etc.}$$

By taking nine terms of this series, we find

$$a = 2.718282,$$

which is the *base* of Napier's system.

427. *The logarithm of a number in any system is equal to the modulus of that system multiplied by the Naperian logarithm of the number.*

If we designate Naperian logarithms by Nap. log., and logarithms in any other system by log., then, since the modulus of Napier's system is unity, we have

$$\log. (1+m) = \text{M}\left(m - \frac{m^2}{2} + \frac{m^3}{3} -, \text{ etc.}\right),$$

$$\text{Nap. log.} (1+m) = m - \frac{m^2}{2} + \frac{m^3}{3} -, \text{ etc.}$$

Hence $\log. (1+m) = \text{M} \times \text{Nap. log.} (1+m),$

or $$\text{M} = \frac{\log. (1+m)}{\text{Nap. log.} (1+m)},$$

where $1+m$ may designate any number whatever.

428. *To render the Logarithmic Series converging.*—The formula of Art. 425,

$$\log. (1+m) = \text{M}\left(m - \frac{m^2}{2} + \frac{m^3}{3} -, \text{ etc.}\right), \qquad (1.)$$

can not be employed for the computation of logarithms when m is greater than unity, because the series does not converge. This series may, however, be transformed into a converging series in the following manner:

Substitute $-m$ for m, and we shall have

$$\log.(1-m)=\mathrm{M}\left(-m-\frac{m^2}{2}-\frac{m^3}{3}-, \text{ etc.}\right). \qquad (2.)$$

Subtracting Eq. (2) from Eq. (1), observing that $\log.(1+m)-\log.(1-m)=\log.\frac{1+m}{1-m}$, we shall have

$$\log.\frac{1+m}{1-m}=2\mathrm{M}\left(m+\frac{m^3}{3}+\frac{m^5}{5}+, \text{ etc.}\right).$$

Now, since this is true for every value of m, put

$$m=\frac{1}{2p+1}, \text{ whence } \frac{1+m}{1-m}=\frac{p+1}{p},$$

and the preceding series, by substitution, becomes

$$\log.\frac{p+1}{p}=\log.(p+1)-\log.p=2\mathrm{M}\left(\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+, \text{ etc.}\right).$$

429. This series converges rapidly, and may be employed for the computation of logarithms in the Naperian or the common systems. It is only necessary to compute the logarithms of *prime* numbers directly, since the logarithm of any other number may be obtained by adding the logarithms of its several factors. Making $p=1, 2, 4, 6$, etc., successively, we obtain the following

Naperian or Hyperbolic Logarithms.

$$\log.2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\ldots\right) \qquad =0.693147$$

$$\log.3=\log.2+2\left(\frac{1}{5}+\frac{1}{3.5^3}+\frac{1}{5.5^5}+\frac{1}{7.5^7}+\ldots\right) \qquad =1.098612$$

$$\log.4=2\log.2 \qquad =1.386294$$

$$\log.5=\log.4+2\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\ldots\right) \qquad =1.609438$$

$$\log.6=\log.3+\log.2 \qquad =1.791759$$

$$\log.7=\log.6+2\left(\frac{1}{13}+\frac{1}{3.13^3}+\frac{1}{5.13^5}+\frac{1}{7.13^7}+\ldots\right)=1.945910$$

$$\log.8=3\log.2 \qquad =2.079442$$

$$\log.9=2\log.3 \qquad =2.197225$$

$$\log.10=\log.5+\log.2 \qquad =2.302585$$

etc., etc., etc.

430. *To construct a Table of Common Logarithms.*—In order to compute logarithms of the common system, we must first determine the value of the modulus. In Art. 427, we found

$$M=\frac{\log.\ (1+m)}{\text{Nap. log. } (1+m)}.$$

If $1+m=a$, the base of the system, then $\log.\ a=1$, and we have

$$M=\frac{1}{\text{Nap. log. } a};$$

that is, *the modulus of any system is the reciprocal of the Naperian logarithm of the base of the system.*

The base of the common system is 10, whose Naperian logarithm is 2.302585. Hence

$$M=\frac{1}{2.302585}=.434294,$$

which is the modulus of the common system.

We can now compute the common logarithms by multiplying the corresponding Naperian logarithms by .434294, Art. 427. In this manner was the table on pages 290–1 computed.

431. *Results.*—The base of Briggs's system is 10.
" Napier's " 2.71828.
The modulus of Briggs's system is 0.43429.
" Napier's " 1.

Since, in Briggs's system, all numbers are to be regarded as powers of 10, we have

$$10^{0.301}=2,$$
$$10^{0.477}=3,$$
$$10^{0.602}=4, \text{ etc.}$$

In Napier's system, all numbers are to be regarded as powers of 2.71828. Thus,

$$2.718^{0.693}=2,$$
$$2.718^{1.098}=3,$$
$$2.718^{1.386}=4, \text{ etc.}$$

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CHAPTER XXI.

GENERAL THEORY OF EQUATIONS.

432. A *cubic equation* with one unknown quantity is an equation in which the highest power of this quantity is of the third degree, as, for example, $x^3-6x^2+8x-15=0$. All equations of the third degree with one unknown quantity may be reduced to the form

$$x^3+ax^2+bx+c=0.$$

A *biquadratic equation* with one unknown quantity is an equation in which the highest power of this quantity is of the fourth degree, as, for example, $x^4-6x^3+7x^2+5x-4=0$. Every equation of the fourth degree with one unknown quantity may be reduced to the form

$$x^4+ax^3+bx^2+cx+d=0.$$

The general form of an equation of the fifth degree with one unknown quantity is

$$x^5+ax^4+bx^3+cx^2+dx+e=0;$$

and the general form of an equation of the nth degree with one unknown quantity is

$$x^n+\mathrm{A}x^{n-1}+\mathrm{B}x^{n-2}+\mathrm{C}x^{n-3}+\ldots\ldots+\mathrm{T}x+\mathrm{V}=0. \qquad (1.)$$

This equation will be frequently referred to hereafter by the name of the *general equation of the nth degree*, or simply as Equation (1).

An equation not given in this form may be reduced to it by transposing all the terms to the first member, arranging them according to the descending powers of the unknown quantity, and dividing by the coefficient of the first term. In this equation n is a positive whole number, but the coefficients A, B, C, etc., may be either positive or negative, entire or fractional, rational or irrational, real or imaginary. The term V may be regarded as the coefficient of x^0, and is called the *absolute* term of the equation.

It is obvious that if we could solve this equation we should have the solution of every equation that could be proposed. Unfortunately, no general solution has ever been discovered; yet many important properties are known which enable us to solve any numerical equation.

433. Any expression, either numerical or algebraic, real or imaginary, which, being substituted for x in Equation (1), will satisfy it, that is, make the two members equal, is called *a root of the equation.*

It is assumed that Eq. (1) has at least *one root;* for, since the first member is equal to zero, it will be so for some value of x, either real or imaginary, and this value of x is by definition a root.

434. *If a is a root of the general equation of the nth degree, its first member can be exactly divided by $x-a$.*

For we may divide the first member by $x-a$, according to the usual rule for division, and continue the operation until a remainder is found which does not contain x. Let Q denote the quotient, and R the remainder, if there be one. Then we shall have

$$x^n+Ax^{n-1}+Bx^{n-2}+\dots\quad+Tx+V=Q(x-a)+R. \quad (2.)$$

Now, if a is a root of the proposed equation, it will reduce the first member of (2) to 0; it will also reduce $Q(x-a)$ to 0; hence R is also equal to 0. But, by hypothesis, R does not contain x; it is therefore equal to 0, whatever value be attributed to x, and, consequently, the first member is exactly divisible by $x-a$.

435. *If the first member of the general equation of the nth degree is exactly divisible by $x-a$, then a is a root of the equation.*

For suppose the division performed, and let Q denote the quotient; then we shall have

$$x^n+Ax^{n-1}+Bx^{n-2}+\dots\quad+Tx+V=Q(x-a).$$

If, in this equation, we make $x=a$, the second member reduces to 0; consequently the first member reduces to 0; and, therefore, a is a root of the equation.

EXAMPLES.

1. Prove that 1 is a root of the equation

$$x^3-6x^2+11x-6=0.$$

The first member is divisible by $x-1$, and gives $x^2-5x+6=0$.

2. Prove that 2 is a root of the equation

$$x^3-x-6=0.$$

The first member is divisible by $x-2$, and gives $x^2+2x+3=0$.

3. Prove that 2 is a root of the equation

$$x^3-11x^2+36x-36=0.$$

4. Prove that 4 is a root of the equation

$$x^3+x^2-34x+56=0.$$

5. Prove that -1 is a root of the equation

$$x^4-38x^3+210x^2+538x+289=0.$$

6. Prove that -5 is a root of the equation

$$x^5+6x^4-10x^3-112x^2-207x-110=0.$$

7. Prove that 3 is a root of the equation

$$x^7+x^6-14x^5-14x^4+49x^3+49x^2-36x-36=0.$$

436. *Every equation of the nth degree containing but one unknown quantity has n roots and no more.*

Since the equation has at least one root, denote that root by a; then will the first member be divisible by $x-a$, and the quotient will be of the form

$$x^{n-1}+A'x^{n-2}+B'x^{n-3}+\ldots\quad +T'x+V',$$

and the given equation may be written under the form

$$(x-a)(x^{n-1}+A'x^{n-2}+\ldots\quad +T'x+V')=0. \qquad (3.)$$

Now equation (3) may be satisfied by supposing either of its factors equal to zero. If the second factor equals zero, we shall have

$$x^{n-1}+A'x^{n-2}+B'x^{n-3}+\ldots\quad +T'x+V'=0. \qquad (4.)$$

Now equation (4) has at least one root; denote that root by b; then will the first member be divisible by $x-b$, and equation (4) can be written under the form

$$(x-b)(x^{n-2}+A''x^{n-3}+\ldots\quad +T''x+V'')=0,$$

which reduces Eq. (3) to the form of

$$(x-a)(x-b)(x^{n-2}+A''x^{n-3}+\ldots\quad T''x+V'')=0.$$

By continuing this process, it may be shown that the first member will ultimately be resolved into n binomial factors of the form $x-a$, $x-b$, $x-c$, etc. Hence equation (1) may be written under the form

$$(x-a)(x-b)(x-c)(x-d)\ldots\ (x-k)(x-l)=0. \qquad (5.)$$

This equation may be satisfied by any one of the n values, $x=a$, $x=b$, $x=c$, etc., and, consequently, these values are the roots of the equation.

The equation has *no more* than n roots, because if we ascribe to x a value which is not one of the n values a, b, c, etc., this value will not cause any one of the factors of Eq. (5) to be zero, and the product of several factors can not be zero when neither of the factors is zero.

If both members of Eq. (5) be divided by either of the factors $x-a$, $x-b$, etc., it will be reduced to an equation of the next inferior degree; and if we can depress any equation to a quadratic, its roots can be determined by methods already explained.

Ex. 1. One root of the equation

$$x^3+3x^2-16x+12=0$$

is 1; what are the other roots?

Ex. 2. Two roots of the equation

$$x^4-10x^3+35x^2-50x+24=0$$

are 1 and 3; what are the other roots?

Ex. 3. Two roots of the equation

$$x^4-12x^3+48x^2-68x+15=0$$

are 3 and 5; what are the other roots? *Ans.* $2\pm\sqrt{3}$.

Ex. 4. Two roots of the equation

$$4x^4-14x^3-5x^2+31x+6=0$$

are 2 and 3; what are the other roots? *Ans.* $\dfrac{-3\pm\sqrt{5}}{4}$.

Ex. 5. Two roots of the equation

$$x^4-6x^3+24x-16=0$$

are 2 and -2; what are the other roots? *Ans.* $3\pm\sqrt{5}$.

437. The n roots of an equation of the nth degree are not necessarily all different from each other. *Any number, and, in-*

deed, all of them, may be equal. When we say that an equation of the nth degree has n roots, we simply mean that its first member can be resolved into n binomial factors, equal or unequal, and each factor contains one root.

Thus the equation $x^3-6x^2+12x-8=0$ can be resolved into the factors $(x-2)(x-2)(x-2)=0$, or $(x-2)^3=0$; whence it appears that the three roots of this equation are 2, 2, 2. But, in general, the several roots of an equation differ from each other numerically.

The equation $x^3=8$ has apparently but one root, viz., 2, but by the method of the preceding article we can discover two other roots. Dividing x^3-8 by $x-2$, we obtain $x^2+2x+4=0$. Solving this equation, we find $x=-1\pm\sqrt{-3}$. Thus, the three roots of the equation $x^3=8$ are

$$2;\ -1+\sqrt{-3};\ -1-\sqrt{-3}.$$

The student should verify the last two values by actual multiplication.

Ex. 1. Find the four roots of the equation $x^4-81=0$.

Ex. 2. Find the six roots of the equation $x^6-64=0$.

438. *The coefficient of the second term in the equation of the nth degree is equal to the algebraic sum of the roots with their signs changed.*

The coefficient of the third term is equal to the algebraic sum of the products of all the roots, taken in sets of two.

The coefficient of the fourth term is equal to the algebraic sum of the products of all the roots, taken in sets of three, with their signs changed.

The last term is equal to the continued product of all the roots with their signs changed.

Let $a, b, c, d, \ldots\ l$, represent the roots of an equation of the nth degree. This equation will accordingly contain the factors $x-a$, $x-b$, etc.; that is, we shall have

$$(x-a)(x-b)(x-c)(x-d)\ldots\ (x-l)=0.$$

If we perform the multiplication as in Art. 351, we shall have

$$\begin{array}{r|c|c|l}
x^n-a & x^{n-1}+ab & x^{n-2}-abc & x^{n-3}+\ldots\ldots \quad -(abcd\ldots\ldots \; l)=0; \\
-b & +ac & -abd & \\
-c & +ad & -acd & \\
-d & +bd & -bcd & \\
\text{etc.} & +cd & \text{etc.} & \\
 & \text{etc.} & &
\end{array}$$

which results are seen to conform to the laws above stated. By the method employed in Art. 352 it may be proved that if these laws hold true for the product of n binomial factors, they will also hold true for the product of $n+1$ binomial factors. But we have found by actual multiplication that these laws are true for the product of four factors, hence they are true for the product of five factors. Being true for five, they must be true for six, and so on for any number of factors.

It will be perceived that these properties include those of quadratic equations mentioned on pages 203–5.

If the roots are all negative, the signs of all the terms of the equation will be positive, because all the signs of the factors of which the equation is composed are positive.

If the roots are all positive, the signs of the terms will be alternately positive and negative.

If the sum of the positive roots is numerically equal to the sum of the negative roots, their algebraic sum will be zero; consequently the coefficient of the second term of the equation will be zero, and that term will disappear from the equation. Conversely, if the second term of the equation is wanting, the sum of the positive roots is numerically equal to the sum of the negative roots.

Ex. 1. Form the equation whose roots are 1, 2, and 3.

For this purpose we must multiply together the factors $x-1$, $x-2$, $x-3$, and we obtain $x^3-6x^2+11x-6=0$.

This example conforms to the rules above given for the coefficients. Thus the coefficient of the second term is equal to the sum of all the roots, $1+2+3$, with their signs changed.

The coefficient of the third term is the sum of the products of the roots taken two and two; thus,

$$1\times2+1\times3+2\times3=11.$$

The last term is the product of all the roots, $1\times2\times3$, with their signs changed.

Ex. 2. Form the equation whose roots are 2, 3, 5, and -6.

Ans. $x^4-4x^3-29x^2+156x-180=0$.

Show how these coefficients conform to the laws above given.

Ex. 3. Form the equation whose roots are

$$1,\ 1,\ 1,\ -1, \text{ and } -2.$$

Ex. 4. Form the equation whose roots are

$$1,\ 3,\ 5,\ -2,\ -4, \text{ and } -6.$$

Ans. $x^6+3x^5-41x^4-87x^3+400x^2-444x-720=0$.

Ex. 5. Form the equation whose roots are

$$1\pm\sqrt{-2} \text{ and } 2\pm\sqrt{-3}.$$

Ex. 6. Form the equation whose roots are

$$1\pm\sqrt{-1} \text{ and } 2\pm\sqrt{3}.$$

439. Since the last term is the continued product of all the roots of an equation, *it must be exactly divisible by each of them.*

For example, take the equation $x^3-x-6=0$. Its roots must all be divisors of the last term, 6; hence, if the equation has a rational root, it must be one of the numbers 1, 2, 3, or 6, either positive or negative; and, by trial, we can easily ascertain whether either of these numbers will satisfy the equation. We thus find that $+2$ is one of the roots, and, by the method of Art. 436, we find the remaining roots to be $-1\pm\sqrt{-2}$.

If the last term of an equation vanishes, as in the example $x^4+2x^3+3x^2+6x=0$, the equation is divisible by $x-0$, and consequently 0 is one of its roots. If the last two terms vanish, then two of its roots are equal to zero.

440. *If the coefficients of an equation are whole numbers, and the coefficient of its first term unity, the equation can not have a root which is a rational fraction.*

Suppose, if possible, that $\frac{a}{b}$ is a root of the general equation of the nth degree, where $\frac{a}{b}$ represents a rational fraction expressed in its lowest terms. Substitute this value for x in the given equation, and we have

$$\frac{a^n}{b^n}+A\frac{a^{n-1}}{b^{n-1}}+B\frac{a^{n-2}}{b^{n-2}}+\ldots\ldots+T\frac{a}{b}+V=0.$$

Multiplying each term by b^{n-1}, and transposing, we obtain

$$\frac{a^n}{b}=-(Aa^{n-1}+Ba^{n-2}b+\ldots+Tab^{n-2}+Vb^{n-1}).$$

Now, by supposition, a, b, A, B, C, etc., are whole numbers; hence the right-hand member of the equation is a whole number.

But, by hypothesis, $\frac{a}{b}$ is an irreducible fraction; that is, a and b contain no common factor. Consequently, a^n and b will contain no common factor; that is, $\frac{a^n}{b}$ is a fraction in its lowest terms. Hence the supposition that the irreducible fraction $\frac{a}{b}$ is a root of the equation leads to this absurdity, that an irreducible fraction is equal to a whole number.

This proposition only asserts that *every commensurable root must be an integer.* The roots can not be of the form of $\frac{3}{4}$, $\frac{5}{7}$, $\frac{9}{8}$, etc. The equation may have other roots which are incommensurable or imaginary, as $2\pm\sqrt{3}$, $1\pm\sqrt{-2}$.

441. *Any equation having fractional coefficients can be transformed into another which has all its coefficients integers, and the coefficient of its first term unity.*

Reduce the equation to the form

$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots\ldots+Tx+V=0,$$

in which A, B, C, etc., are all integers, either positive or negative.

Substitute for x the value $x=\frac{y}{A}$,

and the equation becomes

$$\frac{y^n}{A^{n-1}}+\frac{By^{n-1}}{A^{n-1}}+\frac{Cy^{n-2}}{A^{n-2}}+\ldots\ldots+\frac{Ty}{A}+V=0;$$

which, multiplied by A^{n-1}, becomes

$$y^n+By^{n-1}+ACy^{n-2}+\ldots\ldots+A^{n-2}Ty+A^{n-1}V=0,$$

in which the coefficients are all integers, and that of the first term unity.

The substitution of $\frac{y}{A}$ for x is not always the one which leads to the most simple result; but when A contains two or more equal factors, each factor need scarcely ever be repeated more than once.

Ex. 1. Transform the equation $x^3-\frac{3x^2}{2}+\frac{5x}{4}-\frac{2}{9}=0$ into another whose coefficients are integers, and that of the first term unity.

Clearing of fractions, we have

$$36x^3-54x^2+45x-8=0.$$

Substituting $\frac{y}{6}$ for x, the transformed equation is

$$\frac{y^3}{6}-\frac{9y^2}{6}+\frac{45y}{6}-8=0,$$

or

$$y^3-9y^2+45y-48=0.$$

Transform the following equations into others whose coefficients are integers, and that of the first term unity.

Ex. 2. $x^3+2x^2+\frac{x}{4}-\frac{1}{9}=0.$ *Ans.* $y^3+12y^2+9y-24=0.$

Ex. 3. $x^3+\frac{x^2}{3}-\frac{x}{4}+2=0.$ *Ans.* $y^3+2y^2-9y+432=0.$

Ex. 4. $x^3+2\frac{1}{6}x^2+\frac{x}{6}-\frac{1}{3}=0.$

Ex. 5. $x^4-4\frac{1}{2}x^2+8x+2\frac{1}{16}=0.$

Ex. 6. $x^3-\frac{14x^2}{3}+7x-\frac{10}{3}=0.$

442. *If in any complete equation involving but one unknown quantity the signs of the alternate terms be changed, the signs of all the roots will be changed.*

Take the general equation of the nth degree,

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots=0. \qquad (1.)$$

in which the signs may follow each other in any order whatever.

If we change the signs of the alternate terms, we shall have

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \dots \quad = 0. \qquad (2.)$$

or, changing the sign of every term of the last equation,

$$-x^n + Ax^{n-1} - Bx^{n-2} + Cx^{n-3} - \dots \quad = 0. \qquad (3.)$$

Now, substituting $+a$ for x in equation (1) will give the same result as substituting $-a$ in equation (2), if n be an *even* number; or substituting $-a$ in equation (3), if n be an *odd* number. If, then, a is a root of equation (1), $-a$ will be a root of equation (2), and, of course, a root of equation (3), which is identical with it.

Hence we see that the positive roots may be changed into negative roots, and the reverse, by simply changing the signs of the alternate terms; so that the finding the real roots of any equation is reduced to finding positive roots only.

This rule assumes that the proposed equation is *complete;* that is, that it has all the terms which can occur in an equation of its degree. If the equation be incomplete, we must introduce any missing term with zero for its coefficient.

Ex. 1. The roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$ are 1, 3, and -2; what are the roots of the equation

$$x^3 + 2x^2 - 5x - 6 = 0?$$

Ex. 2. The roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$ are 1, 2, and 3; what are the roots of the equation

$$x^3 + 6x^2 + 11x + 6 = 0?$$

Ex. 3. The roots of the equation $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$ are -1, $+5$, $1+\sqrt{-1}$, and $1-\sqrt{-1}$; what are the roots of the equation $\quad x^4 + 6x^3 + 5x^2 - 2x - 10 = 0?$

443. *If an equation whose coefficients are all real contains imaginary roots, the number of these roots must be even.*

If an equation whose coefficients are all real has a root of the form $a + b\sqrt{-1}$, then will $a - b\sqrt{-1}$ be also a root of the equation. For, let $a + b\sqrt{-1}$ be substituted for x in the equation, the result will consist of a series of terms, of which those involving only the powers of a and the even powers of $b\sqrt{-1}$ will be real, and those which involve the odd powers of $b\sqrt{-1}$ will be imaginary.

If we denote the sum of the real terms by P, and the sum of the imaginary terms by $Q\sqrt{-1}$, the equation becomes

$$P+Q\sqrt{-1}=0.$$

But, according to Art. 243, this equation can only be true when we have separately $P=0$ and $Q=0$.

If we substitute $a-b\sqrt{-1}$ for x in the proposed equation, the result will differ from the preceding only in the signs of the odd powers of $b\sqrt{-1}$, so that the result will be $P-Q\sqrt{-1}$. But we have found that $P=0$ and $Q=0$; hence $P-Q\sqrt{-1}=0$. Therefore $a-b\sqrt{-1}$, when substituted for x, satisfies the equation, and, consequently, it is a *root* of the equation.

It may be proved in a similar manner that if an equation whose coefficients are all *rational*, has a root of the form $a+\sqrt{b}$, then will $a-\sqrt{b}$ be also a root of the equation.

Ex. 1. One root of the equation $x^3-2x+4=0$ is $1+\sqrt{-1}$; what are the other roots?

Ex. 2. One root of the equation $x^3-x^2-7x+15=0$ is $2+\sqrt{-1}$; what are the other roots?

Ex. 3. One root of the equation $x^3-x^2+3x+5=0$ is $1+2\sqrt{-1}$; what are the other roots?

Ex. 4. One root of the equation $x^4-4x^3+4x-1=0$ is $2+\sqrt{3}$; what are the other roots?

Ex. 5. Two roots of the equation

$$x^8+2x^6+4x^5+4x^4-8x^2-16x-32=0$$

are $-1+\sqrt{-1}$ and $1-\sqrt{-3}$; what are the other six roots?

444. *Any equation involving but one unknown quantity may be transformed into another whose roots differ from those of the proposed equation by any given quantity.*

Let it be required to transform the general equation of the nth degree into another whose roots shall be less than those of the proposed equation by a constant difference h.

Assume $y=x-h$, whence $x=y+h$.

Substituting $y+h$ for x in the proposed equation, we have

$$(y+h)^n+A(y+h)^{n-1}+B(y+h)^{n-2}+\ldots\ldots+V=0.$$

Developing the different powers of $y+h$ by the binomial formula, and arranging according to the powers of y, we have

$$\left.\begin{array}{r|r|r|l} y^n+nh & y^{n-1}+\frac{1}{2}n(n-1)h^2 & y^{n-2}+\frac{1}{6}n(n-1)(n-2)h^3 & y^{n-3}+\text{, etc.}\\ +A & +(n-1)Ah & +\frac{1}{2}(n-1)(n-2)Ah^2 & \\ & +B & +(n-2)Bh & \\ & & +C & \end{array}\right\}=0,$$

which equation satisfies the proposed condition, since y is less than x by h. If we assume $y=x+h$, or $x=y-h$, we shall obtain in the same manner an equation whose roots are *greater* than those of the given equation by h.

Ex. 1. Find the equation whose roots are greater by 1 than those of the equation $x^3+3x^2-4x+1=0$.

We must here substitute $y-1$ in place of x.

Ans. $y^3-7y+7=0$.

Ex. 2. Find the equation whose roots are less by 1 than those of the equation $x^3-2x^2+3x-4=0$.

Ans. $y^3+y^2+2y-2=0$.

Ex. 3. Find the equation whose roots are greater by 3 than those of the equation $x^4+9x^3+12x^2-14x=0$.

Ans. $y^4-3y^3-15y^2+49y-12=0$.

Ex. 4. Find the equation whose roots are less by 2 than those of the equation $5x^4-12x^3+3x^2+4x-5=0$.

Ans. $5y^4+28y^3+51y^2+32y-1=0$.

Ex. 5. Find the equation whose roots are greater by 2 than those of the equation $x^5+10x^4+42x^3+86x^2+70x+12=0$.

Ans. $y^5+2y^3-6y^2-10y+8=0$.

445. *Any complete equation may be transformed into another whose second term is wanting.*

Since h in the preceding article may be assumed of any value, we may put $nh+A=0$, which will cause the second term of the general development to disappear. Hence $h=-\frac{A}{n}$, and $x=y-\frac{A}{n}$. Hence, to transform an equation into another which wants the second term, *substitute for the unknown quantity a new unknown quantity minus the coefficient of the second term divided by the highest exponent of the unknown quantity.*

Ex. 1. Transform the equation $x^3-6x^2+8x-2=0$ into another whose second term is wanting.

Put $x=y+2$. *Ans.* $y^3-4y-2=0$.

Ex. 2. Transform the equation $x^4-16x^3-6x+15=0$ into another whose second term is wanting.

Put $x=y+4$. *Ans.* $y^4-96y^2-518y-777=0$.

Ex. 3. Transform the equation

$$x^5+15x^4+12x^3-20x^2+14x-25=0$$

into another whose second term is wanting.

Ans. $y^5-78y^3+412y^2-757y+401=0$.

Ex. 4. Transform the equation $x^4-8x^3+5=0$ into another whose second term is wanting.

According to Art. 438, when the second term of an equation is wanting, the sum of the positive roots is numerically equal to the sum of the negative roots.

446. *If two numbers, substituted for the unknown quantity in an equation, give results with contrary signs, there must be at least one real root included between those numbers.*

Let us denote the real roots of the general equation of the nth degree by a, b, c, etc., and suppose them arranged in the order of their magnitude, a being algebraically the smallest, that is, nearest to $-\infty$; b the next smallest, and so on. The equation may be written under the following form,

$$(x-a)(x-b)(x-c)(x-d)\ldots\ldots=0.$$

Now let us suppose x to increase from $-\infty$ toward $+\infty$, assuming, in succession, every possible value. As long as x is less than a, every factor of the above expression will be negative, and the entire product will be positive or negative according as the number of factors is even or odd. When x becomes equal to a, the whole product becomes equal to 0. But if x be greater than a and less than b, the factor $x-a$ will be positive, while all the other factors will be negative. Hence, when x changes from a value less than a to a value greater than a and less than b, the sign of the whole product changes from + to − or from − to +. When x becomes equal to b, the product again becomes zero; and as x increases from b to c, the factor

$x-b$ becomes positive, and the sign of the product changes again from $-$ to $+$ or from $+$ to $-$; and, in general, the product changes its sign as often as the value of x passes over a real root of the equation.

Hence, if two numbers substituted for x in an equation give results with contrary signs, there must be some intermediate number which reduces the first member to 0, and this number is a root of the equation.

If the two numbers which give results with contrary signs differ from each other only by unity, it is plain that we have found the *integral part* of a root.

If two numbers, substituted for x in an equation, give results with *like* signs, then between these numbers there will either be *no* root, or some *even* number of roots. The last case may include imaginary roots.

For if $a+b\sqrt{-1}$ be a root of the equation, then will $a-b\sqrt{-1}$ be also a root. Now

$$(x-a-b\sqrt{-1})(x-a+b\sqrt{-1})=(x-a)^2+b^2,$$

a result which is always positive; that is, the quadratic factor corresponding to a pair of imaginary roots of an equation whose coefficients are real, is *always positive.*

Ex. 1. Find the first figure of one of the roots of the equation $x^3+x^2+x-100=0$.

When $x=4$, the first member of the equation reduces to -16; and when $x=5$, it reduces to $+55$. Hence there must be a root between 4 and 5; that is, 4 is the first figure of one of the roots.

Ex. 2. Find the first figure of one of the roots of the equation $x^3-6x^2+9x-10=0$.

Ex. 3. Find the first figure of each of the roots of the equation $x^3-4x^2-6x+8=0$.

447. In a series of terms, two successive signs constitute a *permanence* when the signs are alike, and a *variation* when they are unlike. Thus, in the equation $x^3-2x^2-5x+6=0$, the signs of the first two terms constitute a variation, the signs of the second and third constitute a permanence, and those of the third and fourth also a variation.

Descartes's Rule of Signs.

448. *Every equation must have as many variations of sign as it has positive roots, and as many permanences of sign as it has negative roots.*

According to Art. 436, the first member of the general equation of the nth degree may be regarded as the product of n binomial factors of the form $x-a$, $x-b$, etc. The above theorem will then be demonstrated if we prove that the multiplication of a polynomial by a new factor, $x-a$, corresponding to a *positive* root, will introduce at least one *variation*, and that the multiplication by a factor, $x+a$, will introduce at least one *permanence.*

Suppose, for example, that the signs of the terms in the original polynomial are $++---+-+--+$, and we have to multiply the polynomial by a binomial in which the signs of the terms are $+-$. If we write down simply the *signs* which occur in the process and in the result, we have

$$\begin{array}{cccccccccccc}
+ & + & - & - & - & + & - & + & - & - & + & \\
+ & - & & & & & & & & & & \\
\hline
+ & + & - & - & - & + & - & + & - & - & + & \\
& - & - & + & + & + & - & + & - & + & + & - \\
\hline
+ & \pm & - & \pm & \pm & + & - & + & - & \pm & + & -
\end{array}$$

We perceive that the signs in the upper line of the partial products must all be the *same* as in the given polynomial; but those in the lower line are all *contrary* to those of the given polynomial, and advanced one term toward the right. When the corresponding terms of the two partial products have different signs, the sign of that term in the result will depend upon the relative magnitude of the two terms, and may be either $+$ or $-$. Such terms have been indicated by the double sign $\pm$; and it will be observed that the permanences in the given polynomial are changed into signs of ambiguity. Hence, take the ambiguous sign as you will, the permanences in the final product are not increased by the introduction of the positive root $+a$, but the number of signs is increased by *one*, and therefore the number of variations must be increased by *one.* Hence each factor corresponding to a positive root must introduce at least *one new variation*, so that there must be as many variations as there are positive roots.

In the same manner we may prove that the multiplication by a factor, $x+a$, corresponding to a *negative* root, must introduce at least one new *permanence;* so that there must be as many permanences as there are negative roots.

If all the roots of an equation are *real*, the number of positive roots is *equal* to the number of variations, and the number of negative roots is equal to the number of permanences. If the equation is incomplete, we must supply the place of any deficient term with ± 0 before applying the preceding rule.

Ex. 1. The equation $x^5-3x^4-5x^3+15x^2+4x-12=0$ has five real roots; how many of them are positive?

Ex. 2. The equation $x^4-3x^3-15x^2+49x-12=0$ has four real roots; how many of them are negative?

Ex. 3. The equation

$$x^6+3x^5-41x^4-87x^3+400x^2+444x-720=0$$

has six real roots; how many of them are positive?

Derived Polynomials.

449. If we take the general equation of the nth degree, and substitute $y+h$ in place of x, it becomes

$$(y+h)^n+A(y+h)^{n-1}+B(y+h)^{n-2}+\ldots\ldots+V=0.$$

Developing the powers of the binomial $y+h$, and arranging in the order of the powers of h, we have

$$\begin{array}{ll|l|l}
y^n & +ny^{n-1} & h+n(n-1)y^{n-2} & \dfrac{h^2}{1.2}+\ldots\ldots \\
+Ay^{n-1} & +(n-1)Ay^{n-2} & +(n-1)(n-2)Ay^{n-3} & \\
+By^{n-2} & +(n-2)By^{n-3} & +(n-2)(n-3)By^{n-4} & \\
\ldots\ldots & \ldots\ldots & & \\
+Ty & +T & & \\
+V & & &
\end{array}$$

The part of this development which is independent of h is of the same form as the original polynomial, and we will designate it by X. We will denote the coefficient of h by X_1, the coefficient of $\frac{h^2}{1.2}$ by X_2, etc. The preceding development may then be written

$$X+X_1h+X_2\frac{h^2}{1.2}+X_3\frac{h^3}{2.3}+\ldots\quad +h^n.$$

450. The polynomials X_1, X_2, etc., are called *derived polynomials*, or simply *derivatives*. X_1 is called the *first derivative* of X, X_2 the *second derivative*, and so on. X is called the *primitive polynomial. Each derived polynomial is deduced from the preceding by multiplying each term by the exponent of the leading letter in that term, and then diminishing the exponent of the leading letter by unity.*

Ex. 1. What are the successive derivatives of

$$x^3-7x^2+8x-3?$$

Ans. $\begin{cases}\text{1st. } 3x^2-14x+8.\\ \text{2d. } 6x-14.\\ \text{3d. } 6.\end{cases}$

Ex. 2. What are the successive derivatives of

$$x^4-8x^3+14x^2+4x-8?$$

Ex. 3. What are the successive derivatives of

$$x^5+3x^4+2x^3-3x^2-2x-2?$$

Ex. 4. What is the first derivative of

$$x^n+Ax^{n-1}+Bx^{n-2}+\ldots\quad +Tx+V?$$

Equal Roots.

451. We have seen, Art. 436, that if a, b, c, etc., are the roots of the general equation of the nth degree, the equation may be written

$$X=(x-a)(x-b)(x-c)\ldots\quad (x-k)(x-l)=0.$$

When the equation has two roots equal to a, there will be two factors equal to $x-a$; that is, the first member will be divisible by $(x-a)^2$; when there are three roots equal to a, the first member will be divisible by $(x-a)^3$; and if there are n roots equal to a, the first member will contain the factor $(x-a)^n$. The first derivative will contain the factor $n(x-a)^{n-1}$; that is,

$x-a$ occurs $(n-1)$ times as a factor in the first derivative. The greatest common divisor of the primitive polynomial, and its first derivative, must therefore contain the factor $x-a$, repeated once less than in the primitive polynomial.

Hence, to determine whether an equation has equal roots, we have the following

RULE.

Find the greatest common divisor between the given polynomial and its first derivative. If there is no common divisor the equation has no equal roots. If there is a common divisor, place this equal to zero, and solve the resulting equation.

Ex. 1. Find the equal roots of the equation

$$x^3-8x^2+21x-18=0.$$

The first derivative is $3x^2-16x+21$.

The greatest common divisor between this and the given polynomial is $x-3$.

Hence the equation has two roots, each equal to 3.

Ex. 2. Find the equal roots of the equation

$$x^3-13x^2+55x-75=0.$$

Ans. Two roots equal to 5.

Ex. 3. Find the equal roots of the equation

$$x^3-7x^2+16x-12=0.$$

Ans. Two roots equal to 2.

Ex. 4. Find the equal roots of the equation

$$x^4-6x^2-8x-3=0.$$

Ans. Two roots equal to -1.

Ex. 5. Find the equal roots of the equation

$$x^3-3x^2-9x+27=0.$$

Ex. 6. Find the equal roots of the equation

$$x^3+8x^2+20x+16=0.$$

Sturm's Theorem.

452. *The object of Sturm's theorem* is to determine the number of the real roots of an equation, and likewise the situation of these roots, or their initial figures when the roots are irrational.

According to Art. 446, if we suppose x to assume in succession every possible value from $-\infty$ to $+\infty$, and determine the number of times that the first member of the equation changes its sign, we shall have the number of real roots, and, consequently, the number of imaginary roots in the equation, since the real and imaginary roots are together equal in number to the degree of the equation. Sturm's theorem enables us easily to determine the number of such changes of sign.

453. *Sturm's Functions.*—Let the first member of the general equation of the nth degree, after having been freed from its equal roots, be denoted by X, and let its first derivative be denoted by X_1. We now apply to X and X_1 the process of finding their greatest common divisor, with this modification, that we change the sign of each remainder before taking it as a divisor; that is, divide X by X_1, and denote the remainder with its sign changed by R; also, divide X_1 by R, and denote the remainder with its sign changed by R_1, and so on to R_n, which will be a numerical remainder independent of x, since, by hypothesis, the equation $X=0$ has no equal roots.

We thus obtain the series of quantities

$$X, X_1, R, R_1, R_2, \ldots \; R_n,$$

each of which is of a lower degree with respect to x than the preceding; and the last is altogether independent of x, that is, does not contain x.

We now substitute for x in the above functions any two numbers, p and q, of which p is less than q. The substitution of p will give results either positive or negative. If we only take account of the *signs* of the results, we shall obtain a certain number of *variations* and a certain number of *permanences.*

The substitution of q for x will give a second series of signs, presenting a certain number of variations and permanences. The following, then, is *the Theorem of Sturm.*

454. *If, in the series of functions* $X, X_1, R, R_1, \ldots \; R_n$, *we substitute in place of x any two numbers, p and q, either positive or negative, and note the signs of the results, the difference between the number of variations of sign when $x=p$ and when $x=q$ is equal* *to the number of real roots of the equation* $X=0$ *comprised between* p *and* q.

Let $Q, Q_1, Q_2, \ldots Q_n$, denote the quotients in the successive divisions. Now, since the dividend is equal to the product of the divisor and quotient *plus* the remainder, or *minus* the remainder with its sign changed, we must have the following equations:

$$X=X_1 Q-R, \qquad (1.)$$
$$X_1=RQ_1-R_1, \qquad (2.)$$
$$R=R_1 Q_2-R_2, \qquad (3.)$$
$$\ldots\ldots \quad \ldots$$
$$R_{n-2}=R_{n-1} Q_n-R_n. \qquad (n-1).$$

From these equations we deduce the following conclusions:

455. *If, in the series of functions* X, X_1, R, etc., *any number be substituted for* x, *two consecutive functions can not reduce to zero at the same time.*

For, if possible, suppose $X_1=0$ and $R=0$; then, by Eq. (2), we shall have $R_1=0$. Also, since $R=0$ and $R_1=0$, by Eq. (3) we must have $R_2=0$; and from the next equation $R_3=0$, and so on to the last equation, which will give $R_n=0$, which is impossible, since it was shown that this final remainder is independent of x, and must therefore remain unchanged for every value of x.

456. *When, by the substitution of any number for* x, *any one of these functions becomes zero, the two adjacent functions must have contrary signs for the same value of* x.

For, suppose R_1 in Eq. (3) becomes equal to zero, then this equation will reduce to $R=-R_2$; that is, R and R_2 have contrary signs.

457. *If* a *is a root of the equation* $X=0$, *the signs of* X *and* X_1 *will constitute a variation for a value of* x *which is a little less than* a, *and a permanence for a value of* x *which is a little greater than* a.

Let h denote a positive quantity as small as we please, and let us substitute $a+h$ for x in the equation $X=0$. According to Art. 449, the development will be of the form

$$X+X_1h+X_2\frac{h^2}{2}+\text{other terms involving higher powers of } h.$$

Now, if a is a root of the proposed equation, it must reduce the polynomial X to zero, and the development becomes

$$X_1h+X_2\frac{h^2}{2}+\text{other terms involving higher powers of } h,$$

or $$h(X_1+X_2\frac{h}{2}+, \text{ etc.}).$$

Also, if we substitute $a+h$ for x in the first derived polynomial, the development will be of the form

$$X_1+X_2h+\text{other terms involving higher powers of } h.$$

Now a value may be assigned to h so small that the first term of each of these developments shall be greater than the sum of all the subsequent terms. For if h be made indefinitely small, then will X_2h be indefinitely small in comparison with X_1, which is finite; and, since the following terms contain higher powers of h than the first, each will be indefinitely small in comparison with the preceding term; and, since the number of terms is finite, the first term must be greater than the sum of the subsequent terms. Hence, when h is taken indefinitely small, the sum of the terms of the two developments must have the same sign as their first terms,

$$X_1h \text{ and } X_1.$$

When h is positive, these terms must both have the *same* sign; and when h is negative, they must have *contrary* signs; that is, the signs of the two functions X and X_1 constitute a variation when $x=a-h$, and a permanence when $x=a+h$.

458. *Demonstration of Sturm's Theorem.*—Suppose all the real roots of the equations

$$X=0,\ X_1=0,\ R=0,\ R_1=0, \text{ etc.},$$

to be arranged in a series in the order of magnitude, beginning with the least. Let p be less than the least of these roots, and let it increase continually until it becomes equal to q, which we suppose to be greater than the greatest of these roots. Now, so long as p is *less* than any of the roots, no change of sign will occur from the substitution of p for x in any of these functions, Art. 446. But suppose p to pass from a number a little smaller to a number a little greater than a root of the equation $X=0$, the sign of X will be changed from + to — or from — to +, Art. 446. The signs of X and X_1 constitute a variation before the change, and a permanence after the change, Art. 457; that is, there is a variation lost or changed into a permanence.

Again, while p increases from a number a little smaller to a number a little greater than another root of the equation $X=0$, a second variation will be changed into a permanence, and so on for the other roots of the given equation.

But when p arrives at a root of any of the other functions X_1, R, R_1, its substitution for x reduces that polynomial to zero, and neither the preceding nor succeeding functions can vanish for the same value of x, Art. 455; and these two adjacent functions have contrary signs, Art. 456. Hence the entire number of variations of sign is not affected by the vanishing of any function intermediate between X and R_n, for the three adjacent functions must reduce to $+0-$ or $-0+$.

Here is one variation, and there will also be one variation if we supply the place of the 0 with either + or —; thus, $+\pm-$ or $-\pm+$.

Thus we have proved that during all the changes of p, Sturm's functions never lose a variation except when p passes through a root of the equation $X=0$, and they never gain a variation. Hence the number of variations lost while x increases from p to q is equal to the number of the roots of the equation $X=0$, which lie between p and q.

Now, since all the real roots must be comprised within the limits $-\infty$ and $+\infty$, if we substitute these values for x in the series of functions X, X_1, etc., the number of variations lost will indicate the whole number of real roots. A third supposition that $x=0$ will show how many of these roots are positive and how many negative; and if we wish to determine smaller limits of the roots, we must try other numbers. It is generally best in the first instance to make trial of such numbers as are most convenient in computation, as 1, 2, 10, etc.

EXAMPLES.

1. Find the number and situation of the real roots of the equation $x^3-3x^2-4x+13=0$.

Here we have $X=x^3-3x^2-4x+13$, and $X_1=3x^2-6x-4$. Dividing X by X_1, we find for a remainder $-14x+35$. Rejecting the factor 7, and changing the sign of the result, we have $R=2x-5$. Multiplying X_1 by 4, and dividing by R, we find for a remainder -1. Changing the sign, we have $R_1=+1$.

Hence we have

$$X=x^3-3x^2-4x+13,$$
$$X_1=3x^2-6x-4,$$
$$R=2x-5,$$
$$R_1=+1.$$

If we substitute $-\infty$ for x in the polynomial X, the sign of the result is $-$; if we substitute $-\infty$ for x in the polynomial X_1, the sign of the result is $+$; if we substitute $-\infty$ for x in the expression $2x-5$, the sign of the result is $-$; and R_1, being independent of x, will remain $+$ for every value of x, so that, by supposing $x=-\infty$, we obtain the series of signs

$$- + - +.$$

Proceeding in the same manner for other assumed values of x, we shall obtain the following results:

Assumed Values of x.	Resulting Signs.	Variations.
$-\infty$	$- + - +$	giving 3 variations.
-3	$- + - +$	" 3 "
-2	$+ + - +$	" 2 "
0	$+ - - +$	" 2 "
1	$+ - - +$	" 2 "
2	$+ - - +$	" 2 "
$2\frac{1}{2}$	$- - 0 +$	" 1 "
3	$+ + + +$	" 0 "
$+\infty$	$+ + + +$	" 0 "

We perceive that no change of sign in either function occurs by the substitution for x of any number less than -3; but, in passing from -3 to -2, the function X changes its sign from $-$ to $+$, by which one variation is lost. In passing from 2 to $2\frac{1}{2}$, the function X again changes its sign, and a second variation of sign is lost. Also, in passing from $2\frac{1}{2}$ to 3, the function X again changes its sign, and a third variation is lost; and there are no further changes of sign arising from the substitution of any number between 3 and $+\infty$.

Hence the given equation has 3 real roots; one situated between -2 and -3, one between 2 and $2\frac{1}{2}$, and a third between $2\frac{1}{2}$ and 3. The *initial* figures of the roots are therefore -2, $+2$, and $+2$.

There are *three* changes of sign of the primitive function, *two* of the first derived function, and *one* of the second derived function; but no variation is lost by the change of sign of either of the derived functions; while every change of sign of the primitive function occasions a loss of one variation.

2. Find the number and situation of the real roots of the equation $$x^3-5x^2+8x-1=0.$$

Here we have

$$X=x^3-5x^2+8x-1,$$
$$X_1=3x^2-10x+8,$$
$$R=2x-31,$$
$$R_1=-2295.$$

When $x=-\infty$, the signs are $-+--$, giving 2 variations,
$x=+\infty$, " $+++-$, " 1 "

Hence this equation has but one real root, and, consequently, must have two imaginary roots. Moreover, it is easily proved that the real root lies between 0 and $+1$.

3. Find the number and situation of the real roots of the equation $$x^4-2x^3-7x^2+10x+10=0.$$

Here we have

$$X=x^4-2x^3-7x^2+10x+10,$$
$$X_1=4x^3-6x^2-14x+10, \text{ or } 2x^3-3x^2-7x+5,$$
$$R=17x^2-23x-45,$$
$$R_1=152x-305,$$
$$R_2=+524535.$$

When $x=-\infty$, the signs are $+-+-+$, giving 4 variations,
$x=+\infty$, " $+++++$, " 0 "

Hence the four roots of this equation are real.

Substituting different values for x, we find that

when $x=-3$,	the signs are	$+-+-+$,	giving	4 variations,
$x=-2$,	"	$-++-+$,	"	3 "
$x=-1$,	"	$-++-+$,	"	3 "
$x=\ 0$,	"	$++--+$,	"	2 "
$x=+1$,	"	$+---+$,	"	2 "
$x=+2$,	"	$+---+$,	"	2 "
$x=+2\frac{1}{2}$,	"	$-0-++$,	"	1 "
$x=+3$,	"	$+++++$,	"	0 "

Hence this equation has one negative root between -2 and -3, one negative root between 0 and -1, one positive root between 2 and $2\frac{1}{2}$, and another positive root between $2\frac{1}{2}$ and 3.

4. Find the number and situation of the real roots of the equation $$x^3-7x+7=0.$$

Ans. Three; viz., one between -3 and -4, one between 1 and $1\frac{1}{2}$, and the other between $1\frac{1}{2}$ and 2.

5. Find the number and situation of the real roots of the equation $$2x^4-20x+19=0.$$

Ans. Two; viz., one between -3 and -4, the other between 2 and 3.

6. Find the number and situation of the real roots of the equation $$x^5+2x^4+3x^3+4x^2+5x-20=0.$$

Ans. One, situated between 1 and 2.

7. Find the number and situation of the real roots of the equation $$x^3+3x^2+5x-178=0.$$

Ans. One, situated between 4 and 5.

8. Find the number and situation of the real roots of the equation $$x^4-12x^2+12x-3=0.$$

Ans. Four; viz., one between -3 and -4, one between 0 and $\frac{1}{2}$, one between $\frac{1}{2}$ and 1, and the other between 2 and 3.

9. Find the number and situation of the real roots of the equation $$x^4-8x^3+14x^2+4x-8=0.$$

Ans. Four; viz., one between -1 and 0, one between 0 and $+1$, one between 3 and 4, and the other between 5 and 6.

Solution of Simultaneous Equations of any Degree.

459. One of the most general methods for the elimination of unknown quantities from a system of equations, depends upon *the principle of the greatest common divisor.*

Suppose we have two equations involving x and y. We first transpose all the terms to one member, so that the equations will be of the form

$$A=0,\ B=0.$$

We arrange the terms in the order of the powers of x, and we will suppose that the polynomial B is not of a higher degree than A. We divide A by B, as in the method of finding the greatest common divisor, Art. 95, and continue the operation as far as possible without introducing fractional quotients having x in the denominator. Let Q represent the quotient, and R the remainder; we shall then have

$$A=BQ+R.$$

But, since A and B are each equal to zero, it follows that R must be equal to zero. If, then, there are certain values of x and y which render A and B equal to zero, these values should be the roots of the equations

$$B=0,\ R=0.$$

We now divide B by R, and continue the operation as far as possible without introducing fractional quotients having x in the denominator. Let R' denote the remainder after this division. For the same reason as before, R' must equal zero, and we thus obtain the two equations

$$R=0,\ R'=0,$$

whose roots must satisfy the equations $A=0$, $B=0$. If we continue to divide each remainder by the succeeding, and suppose that each remainder is of a lower degree with respect to x than the divisor, we shall at last obtain a remainder which does not contain x. Let R'' denote this remainder. The equation $R''=0$ will furnish the values of y, and the equation $R'=0$ will furnish the corresponding values of x.

If we have three equations involving three unknown quantities, we commence by reducing them to two equations with two unknown quantities, and subsequently to a single final equation by a process similar to that above explained.

Ex. 1. Solve the two equations $\begin{cases} x^2+y^2-13=0, \\ x+y-5=0. \end{cases}$

Divide the first polynomial by the second, as follows:

$$\begin{array}{l|l} x^2+y^2-13 & x+y-5 \\ \underline{x^2+(y-5)x} & \overline{x-y+5} \\ \quad -(y-5)x+\ y^2-13 & \\ \quad \underline{-(y-5)x-\ y^2+10y-25} & \\ \qquad\qquad 2y^2-10y+12, \text{ the remainder.} & \end{array}$$

This remainder must be equal to zero; that is,

$$2y^2-10y+12=0,$$

whence $y=2$ or 3.

When $y=2,\ x=3;$

$y=3,\ x=2.$

Ex. 2. Solve the equations $\begin{cases} x+xy^3-18=0, \\ xy+xy^2-12=0. \end{cases}$

Multiply the first polynomial by y, to make its first term divisible, and proceed as follows:

$$\begin{array}{l|l} xy(1+y^3)-18y & xy(1+y)-12 \\ \underline{xy(1+y^3)-12(1-y+y^2)} & \overline{1-y+y^2} \\ \qquad\qquad 12-30y+12y^2, \text{ the remainder.} & \end{array}$$

Hence $12-30y+12y^2=0;$

therefore $y=2$ or $\frac{1}{2}$.

When $y=2,\ x=2,$

$y=\frac{1}{2},\ x=16.$

Ex. 3. Solve the equations $\begin{cases} x^3y^2-x^3+xy+x-6=0, \\ x^3y\ -x^3+x-3=0. \end{cases}$

The first remainder is $3y-3$, which, being placed equal to 0, gives $y=1$, whence $x=3$.

Ex. 4. Solve the equations

$$x^3-3x^2y+x(3y^2-y+1)-y^3+y^2-2y=0,$$
$$x^2-2xy\ +y^2-y=0.$$

The remainder after the first division is $x-2y$, and after the second division y^2-y. Hence we conclude

$$x-2y=0, \text{ and } y^2-y=0.$$

Whence we have $y=1$ or 0,

$x=2$ or 0.

Ex. 5. Solve the equations $\begin{cases} x^2+x(8y-13)+y^2-7y+12=0, \\ x^2-x(4y+1)\ \ +y^2+5y=0. \end{cases}$

The remainder after the first division is

$$x(12y-12)-12y+12.$$

Hence we have $12(y-1)(x-1)=0$, which equation may be satisfied by supposing $y-1=0$ or $x-1=0$.

When $x=1$, $y=-1$ or 0,

$y=1$, $x=\ \ 2$ or 3.

Ex. 6. Solve the equations $\begin{cases} x^3+2x^2y+2xy(y-2)+y^2-4=0. \\ x^2+2xy\ +2y^2-5y+2=0. \end{cases}$

The first division gives a remainder $x(y-2)+y^2-4$, whence we have

$$(y-2)(x+y+2)=0;$$

and we may have either

$$y-2=0, \text{ or } x+y+2=0.$$

If we divide the first member of the second equation by $x+y+2$, we obtain the remainder y^2-5y+6, which also equals zero; whence $y=2$ or 3.

When $y=2$, $x=-4$ or 0,

$y=3$, $x=-5$.

CHAPTER XXII.

SOLUTION OF NUMERICAL EQUATIONS OF HIGHER DEGREES.

461. Equations of the third and fourth degrees can sometimes be solved by direct methods; but these methods are complicated, and are of limited application. No general solution of an equation higher than the fourth degree has yet been discovered. To obtain the roots of numerical equations of degrees higher than the second, we must generally employ tentative methods, or methods which involve approximation.

462. *Commensurable Roots of an Equation.*—Any equation having fractional coefficients can be transformed into another which has all its coefficients integers, and the coefficient of its first term unity, Art. 441, and such an equation can not have a root which is a rational fraction, Art. 440; that is, every commensurable root of this equation must be an integer. Every integral root of this equation is a divisor of the last term, Art. 439. Hence, to find the commensurable roots of an equation, we need only make trial of the integral divisors of the last term.

463. *Method of finding the Roots.*—In order to discover a convenient method of finding the roots, we will form the equation whose roots are 2, 3, 4, and 5. This equation, Art. 436, may be expressed thus,

$$(x-2)(x-3)(x-4)(x-5)=0.$$

If we perform the multiplication here indicated, we shall obtain

$$x^4-14x^3+71x^2-154x+120=0.$$

We know that this equation is divisible by $x-5$, and we will perform the operation by an abridged method. Since the coefficients of the quotient depend simply upon the coefficients of the divisor and dividend, and not upon the literal parts of the terms, we may obtain the coefficients of the quotient by operating upon the coefficients of the divisor and dividend by the usual method. To the coefficients thus found the proper letters may afterward be annexed. The operation may then be exhibited as follows:

$$\begin{array}{llllll}
A & B & C & D & V & \quad r \\
1-14+71-154+120 & & & & & \big|\,1-5,\text{ divisor,} \\
\underline{1-\ \ 5} & & & & & \big|\,1-9+26-24,\text{ quotient.} \\
\quad -\ 9+71 \\
\quad \underline{-\ 9+45} \\
\qquad\quad +26-154 \\
\qquad\quad \underline{+26-130} \\
\qquad\qquad\quad -\ 24+120 \\
\qquad\qquad\quad -\ 24+120.
\end{array}$$

Supplying the powers of x, we obtain for a quotient

$$x^3-9x^2+26x-24=0.$$

In applying this method of division, care should be taken to arrange the terms in the order of the powers of x; and if the series of powers of x in the dividend is incomplete, we must supply the place of the deficient term by a cipher.

The preceding operation may be still further abridged by performing the successive subtractions mentally, and simply writing the results. Represent the root 5 by r, and the coefficients of the given equation by A, B, C, D, ... V.

We first multiply $-r$ by A, and subtract the product from B; the remainder, -9, we multiply by $-r$, and subtract the product from C; the remainder, $+26$, we multiply by $-r$, and subtract the product from D; the remainder, -24, we multiply by $-r$, and, subtracting from V, nothing remains. If we take the root r with a positive sign, we may substitute in the above process addition for subtraction; and if we set down only the successive remainders, the work will be as follows:

$$\begin{array}{l}
A\quad B\quad C\quad D\quad\ V\ \ r \\
1-14+71-154+120(5 \\
1-\ 9+26-\ 24,
\end{array}$$

and the rule will be

Multiply A *by* r, *and add the product to* B; *set down the sum, multiply it by* r, *and add the product to* C; *set down the sum, mul-*

tiply it by r, and add the product to D, *and so on. The final product should be equal to the last term* V, *taken with a contrary sign.*

The coefficients above obtained are the coefficients of a cubic equation whose roots are 2, 3, and 4. The polynomial may therefore be divided by $x-4$, and the operation will be as follows:

$$1-9+26-24(4$$
$$1-5+\ 6.$$

These, again, are the coefficients of a quadratic equation whose roots are 2 and 3. Dividing again by $x-3$, we have

$$1-5+6(3$$
$$1-2,$$

which are the coefficients of the binomial factor $x-2$.

These three operations of division may be exhibited together as follows:

$1-14+71-154+120$	5, first divisor.
$1-\ 9+26-\ 24$	4, second divisor.
$1-\ 5+\ 6$	3, third divisor.
$1-\ 2$	

464. *How to find all the Integral Roots.*—The method here explained will enable us to find all the integral roots of an equation. For this purpose, we make trial of different numbers in succession, all of which must be divisors of the last term of the equation. If any division leaves a remainder, we reject this divisor; if the division leaves no remainder, the divisor employed is a *root* of the equation. Thus, by a few trials, all the integral roots may be easily found.

The labor will often be diminished by first finding positive and negative limits of the roots, for no number need be tried which does not fall within these limits.

Ex. 2. Find the seven roots of the equation

$$x^7+x^6-14x^5-14x^4+49x^3+49x^2-36x-36=0.$$

We take the coefficients separately, as in the last example, and try in succession all the divisors of 36, both positive and negative, rejecting such as leave a remainder. The operation is as follows:

1+1−14−14+49+49−36−36	1, first divisor.
1+2−12−26+23+72+36	2, second divisor.
1+4− 4−34−45−18	3, third divisor.
1+7+17+17+ 6	−1, fourth divisor.
1+6+11+ 6	−1, fifth divisor.
1+5+ 6	−2, sixth divisor.
1+3	−3, seventh divisor.

Hence the seven roots are

$$1,\ 2,\ 3,\ -1,\ -1,\ -2,\ -3.$$

Ex. 3. Find the six roots of the equation

$$x^6+5x^5-81x^4-85x^3+964x^2+780x-1584=0.$$

1+ 5−81− 85+964+ 780−1584	1.
1+ 6−75−160+804+1584	4.
1+10−35−300−396	6.
1+16+61+ 66	− 2.
1+14+33	− 3.
1+11	−11.

The six roots, therefore, are

$$1,\ 4,\ 6,\ -2,\ -3,\ -11.$$

Ex. 4. Find the five roots of the equation

$$x^5+6x^4-10x^3-112x^2-207x-110=0.$$

1+6−10−112−207−110	−1.
1+5−15− 97−110	−2.
1+3−21− 55	−5.
1−2−11	

Three of the roots, therefore, are

$$-1,\ -2,\ -5.$$

The two remainining roots may be found by the ordinary method of quadratic equations. Supplying the letters to the last coefficients, we have

$$x^2-2x-11=0.$$

Hence $\qquad x=1\pm\sqrt{12}.$

Ex. 5. Find the four roots of the equation

$$x^4-12x^3+47x^2-72x+36=0.$$

Ans. 1, 2, 3, and 6.

Ex. 6. Find the four roots of the equation

$$x^4+2x^3-7x^2-8x+12=0.$$

Ex. 7. Find the four roots of the equation

$$x^4-55x^2-30x+504=0.$$

Ex. 8. Find the four roots of the equation

$$x^4-25x^2+60x-36=0.$$

Ex. 9. Find the four roots of the equation

$$x^4-x^3-x^2+19x-42=0.$$

Ex. 10. Find the five roots of the equation

$$x^5+5x^4+x^3-16x^2-20x-16=0.$$

465. *Incommensurable Roots.*—If a high numerical equation is found to contain no commensurable roots, or, if after removing the commensurable roots, the depressed equation is still of a higher degree than the second, we must proceed by approximation to find the incommensurable roots. Different methods may be employed for this purpose; but the following method, which is substantially the same as published by Horner in 1819, is generally to be preferred.

Find, by Sturm's Theorem, or by trial, Art. 446, the integral part of a root, and transform the given equation into another whose roots shall be less than those of the preceding by the number just found, Art. 444. Find, by Art. 446, the first figure of the root of this equation, which will be the first decimal figure of the root of the original equation. Transform the last equation into another whose roots shall be less than those of the preceding by the figure last found. Find, as before, the first figure of the root of this equation, which will be the second decimal figure of the root of the original equation. By proceeding in this manner from one transformation to another, we may discover the successive figures of the root, and may carry the approximation to any degree of accuracy required.

Ex. 1. Find an approximate root of the equation

$$x^3+3x^2+5x=178.$$

We have found, page 330, that this equation has but one real root, and that it lies between 4 and 5. The first figure of the root therefore is 4. Transform this equation into another whose roots shall be less than those of the proposed equation by 4, which is done by substituting $y+4$ for x. We thus obtain

$$y^3+15y^2+77y=46.$$

The first figure of the root of this equation is .5. Transform the last equation into another whose roots shall be less by .5, which is done by substituting $z+.5$ for y. We thus obtain

$$z^3+16.5z^2+92.75z=3.625.$$

The first figure of the root of this equation is .03. Transform the last equation into another whose roots shall be less by .03, which is done by substituting $v+.03$ for z. We thus obtain

$$v^3+16.59v^2+93.7427v=.827623.$$

The first figure of the root of this equation is .008. Transform the last equation into another whose roots shall be less by .008, and thus proceed for any number of figures required.

466. *How the Operation may be abridged.*—This method would be very tedious if we were obliged to deduce the successive equations from each other by the ordinary method of substitution; but they may be derived from each other by a simple law. Thus, let

$$Ax^3+Bx^2+Cx=V \qquad (1.)$$

be any cubic equation, and let the first figure of its root be denoted by r, the second by r', the third by r'', and so on.

If we substitute r for x in equation (1), we shall have

$$Ar^3+Br^2+Cr=V, \textit{ nearly.}$$

Whence
$$r=\frac{V}{C+Br+Ar^2}. \qquad (2.)$$

If we put y for the sum of all the figures of the root except the first, we shall have $x=r+y$; and, substituting this value for x in equation (1), we obtain

$$\left.\begin{array}{l} Ar^3+3Ar^2y+3Ary^2+Ay^3 \\ \quad +\ Br^2+2Bry\ +By^2 \\ \qquad\qquad +Cr\quad\ +Cy \end{array}\right\}=V;$$

or, arranging according to the powers of y, we have

$$Ay^3+(B+3Ar)y^2+(C+2Br+3Ar^2)y=V-Cr-Br^2-Ar^3.$$

Let us put B' for the coefficient of y^2, C' for the coefficient of y, and V' for the right member of the equation, and we have

$$Ay^3+B'y^2+C'y=V'. \qquad (3.)$$

This equation is of the same form as equation (1); and, proceeding in the same manner, we shall find

$$r'=\frac{V'}{C'+B'r'+Ar'^2}, \qquad (4.)$$

where r' is the first figure of the root of equation (3), or the second figure of the root of equation (1).

Putting z for the sum of all the remaining figures, we have $y=r'+z$; and, substituting this value in equation (3), we shall obtain a new equation of the same form, which may be written

$$Az^3+B''z^2+C''z=V''; \qquad (5.)$$

and in the same manner we may proceed with the remaining figures.

Equation (2) furnishes the value of the first figure of the root; equation (4) the second figure, and similar equations would furnish the remaining figures. Each of these expressions involves the unknown quantity which is sought, and might therefore appear to be useless in practice. When, however, the root has been found to several decimal places, the value of the terms Br and Ar^2 will be very small compared with C, and r will be very nearly equal to $\frac{V}{c}$. We may therefore employ C as an *approximate divisor*, which will probably furnish a new figure of the root. Thus, in the last example, all the figures of the root after the first are found by division.

$$46\div77 \quad =.5,$$
$$3.62\div92.75=.03,$$
$$.827\div93.74=.008.$$

If we multiply the first coefficient A by r, the first figure of the root, and add the product to the second coefficient, we shall have

$$B+Ar. \qquad (6.)$$

If we multiply expression (6) by r, and add the product to the third coefficient, we shall have

$$C+Br+Ar^2. \qquad (7.)$$

If we multiply expression (7) by r, and subtract the product from V, we shall have

$$V-Cr-Br^2-Ar^3,$$

which is the quantity represented by V′ *in equation* (3).

If we multiply the first coefficient A by r, and add the product to expression (6), we shall have

$$B+2Ar. \qquad (8.)$$

If we multiply expression (8) by r, and add the product to expression (7), we shall have

$$C+2Br+3Ar^2,$$

which is the coefficient of y in equation (3).

If we multiply the first coefficient A by r, and add the product to expression (8), we shall have

$$B+3Ar,$$

which is the coefficient of y^2 in equation (3).

We have thus obtained the coefficients of the first transformed equation; and, by operating in the same manner upon these coefficients, we shall obtain the coefficients of the second transformed equation, and so on; and the successive figures of the root are indicated by dividing V by C, V′ by C′, V″ by C″, etc.

467. The results of the preceding discussion are expressed in the following

RULE.

Represent the coefficients of the different terms by A, B, C, *and the right-hand member of the equation by* V. *Having found r, the first figure of the root, multiply* A *by r, and add the product to* B. *Set down the sum under* B; *multiply this sum by r, and add the product to* C. *Set down the sum under* C; *multiply it by r, and subtract the product from* V; *the remainder will be the* FIRST DIVIDEND.

Again, multiply A *by r, and add the product to the last number under* B. *Multiply this sum by r, and add the product to the last number under* C; *this result will be the* FIRST TRIAL DIVISOR.

Again, multiply A *by r, and add the product to the last number under* B.

Find the second figure of the root by dividing the first dividend by the first trial divisor, and proceed with this second figure precisely as was done with the first figure, carefully regarding the local value of the figures.

The second figure of the root obtained by division will freqently furnish a result too large to be subtracted from the remainder V′, in which case we must assume a different figure. After the second figure of the root has been obtained, there will seldom be any further uncertainty of this kind.

It may happen that one of the trial divisors becomes zero. In this case equation (2) becomes

$$r=\frac{V}{Br+Ar^2}, \text{ or } \frac{V}{Br} \text{ nearly};$$

whence

$$r^2=\frac{V}{B}, \text{ or } r=\sqrt{\frac{V}{B}};$$

that is, the next figure of the root will be indicated by dividing the last dividend by the last number under B, and extracting the square root of the quotient.

The entire operation for finding a root of the equation

$$x^3+3x^2+5x=178$$

may be exhibited as follows:

A	B	C	V	r
1	+3	+5	=178	(4.5388=x.
	4	28	132	
	7	33	46 =1st dividend.	
	4	44	42.375	
	11	77=1st divisor.	3.625=2d dividend.	
	4	7.75	2.797377	
	15.5	84.75	.827623=3d dividend.	
	.5	8.00	.751003872	
	16.0	92.75=2d divisor.	.076619128=4th dividend.	
	.5	.4959		
	16.53	93.2459		
	3	.4968		
	16.56	93.7427=3d divisor.		
	3	.132784		
	16.598	93.875484		
	8	.132848		
	16.606	94.008332=4th divisor.		

Having found one root, we may depress the equation

$$x^3+3x^2+5x-178=0$$

to a quadratic by dividing it by $x-4.5388$. We thus obtain

$$x^2+7.5388x+39.2173=0,$$

where x is evidently imaginary, because q is negative and greater than $\frac{p^2}{4}$. See Art. 280.

After thus obtaining the root to five or six decimal places, *several more* figures will be correctly obtained by simply dividing the last dividend by the last divisor.

Ex. 2. Find all the roots of the equation

$$x^3+11x^2-102x=-181.$$

The first figure of one of the roots we readily find to be 3. We then proceed, according to the Rule, to obtain the root to four decimal places, after which two more will be obtained correctly by division.

A	B	C	V	r
1	+11	−102	= −181	(3.21312 = x.
	3	42	−180	
	14	−60	−1 = 1st dividend.	
	3	51	−.992	
	17	−9 = 1st divisor.	−.008 = 2d dividend.	
	3	4.04	−.006739	
	20.2	−4.96	−.001261 = 3d dividend.	
	2	4.08	−.001217403	
	20.4	−0.88 = 2d divisor.	−.000043597 = 4th dividend.	
	2	.2061		
	20.61	−.6739		
	1	.2062		
	20.62	−.4677 = 3d divisor.		
	1	.061899		
	20.633	−.405801		
	3	.061908		
	20.636	−.343893 = 4th divisor.		

The two remaining roots may be found in the same way, or by depressing the original equation to a quadratic. Those roots are,

3.22952

−17.44265.

When a power of x is wanting in the proposed equation, we must supply its place with a cipher.

Ex. 3. Find all the roots of the cubic equation

$$x^3-7x=-7.$$

The work of the following example is exhibited in an abbreviated form. Thus, when we multiply A by r, and add the product to B, we set down simply this *result.* We do the same in the next column, thus dispensing with half the number of lines employed in the preceding example. Moreover, we may omit the ciphers on the left of the successive dividends, if we pay proper attention to the local value of the figures. Thus it will be seen that in the operation for finding each successive figure of the root, the decimals under B increase *one* place, those under C increase *two* places, and those under V increase *three* places.

1+0	−7	=−7	(1.356895867=x.
1	−6	−6	
2	−4=1st div'r.	−1=1st dividend.	
3.3	−3.01	−.903	
3.6	−1.93=2d div'r.	−97=2d dividend.	
3.95	−1.7325	86625	
4.00	−1.5325=3d div'r.	10375=3d dividend.	
4.056	−1.508164	9048984	
4.062	−1.483792=4th div'r.	1326016=4th dividend.	
4.0688	−1.48053696	1184429568	
4.0696	−1.47728128=5th div.	141586432=5th div'd.	
4.07049	−1.4769149359	132922344231	
4.07058	−1.4765485837=6th div.	8664087769=6th div'd.	

Having proceeded thus far, four more figures of the root, 5867, are found by dividing the sixth dividend by the sixth divisor.

We may find the two remaining roots by the same process; or, after having obtained one root, we may depress the equation

$$x^3-7x+7=0$$

to a quadratic equation by dividing by $x-1.356895867$, and we shall obtain

$$x^2+1.356895867x-5.158833606=0.$$

Solving this equation, we obtain

$$x = -.678447933 \pm \sqrt{5.619125204}.$$

Hence the three roots are . . . . $\begin{cases} -3.048917, \\ 1.356896, \\ 1.692021. \end{cases}$

Ex. 4. Find a root of the equation $2x^3 + 3x^2 = 850$.

2+3	+0	=850 (7.0502562208
17	119	833
31	336 = 1st divisor.	17 = 1st dividend.
45.10	338.2550	16.912750
45.20	340.5150 = 2d divisor.	.87250 = 2d dividend.
45.3004	340.52406008	68104812016
45.3008	340.53312024 = 3d div.	19145187984 = 3d div'd.
45.30130	340.5353853050	17026769265250
45.30140	340.5376503750 = 4th d.	2118418718750 = 4th div.

Dividing the fourth dividend by the fourth divisor, we obtain the figures 62208, which make the root correct to the tenth decimal place.

The two remaining values of x may be easily shown to be imaginary.

When a negative root is to be found, we change the signs of the alternate terms of the equation, *Art.* 442, and proceed as for a positive root.

Ex. 5. Find a root of the equation $5x^3 - 6x^2 + 3x = -85$.

Changing the signs of the alternate terms, it becomes

$$5x^3 + 6x^2 + 3x = +85.$$

5 +6	+3	+85 (2.16139.
16	35	70
26	87 = 1st divisor.	15 = 1st dividend.
36.5	90.65	9.065
37.0	94.35 = 2d divisor.	5.935 = 2d dividend.
37.80	96.6180	5.797080
38.10	98.9040 = 3d divisor.	137920 = 3d dividend.
38.405	98.942405	98942405
38.410	98.980815 = 4th divisor.	38977595 = 4th div'd.
38.4165	98.99233995	29697701985
38.4180	99.00386535 = 5th div'r.	9279893015 = 5th div'd.

Hence one root of the equation

$$5x^3-6x^2+3x=-85$$

is -2.16139.

The same method is applicable to the extraction of the cube root of numbers.

Ex. 6. Let it be required to extract the cube root of 9; in other words, it is required to find a root of the equation

$$x^3=9.$$

1+0	+0	=9	(2.0800838.
2	4	8	
4	12=1st divisor.	1=1st dividend.	
6.08	12.4864	.998912	
6.16	12.9792=2d divisor.	1088=2d dividend.	
6.24008	12.9796992064	1038375936512	
6.24016	12.9801984192=3d d.	49624063488=3d div.	
6.240243	12.980217139929	38940651419787	
6.240246	12.980235860667=4th d.	10683412068213=4th d.	

Ex. 7. Find all the roots of the equation

$$x^3-15x^2+63x-50=0.$$

Ans. $\begin{cases} 1.02804. \\ 6.57653. \\ 7.39543. \end{cases}$

Ex. 8. Find all the roots of the equation

$$x^3+9x^2+24x+17=0.$$

Ans. $\begin{cases} -1.12061. \\ -3.34730. \\ -4.53209. \end{cases}$

Ex. 9. Extract the cube root of 48228544.

Ans. 364.

Ex. 10. There are two numbers whose difference is 2, and whose product, multiplied by their sum, makes 100. What are those numbers?

Ex. 11. Find two numbers whose difference is 6, and such that their sum, multiplied by the difference of their cubes, may produce 5000.

Ex. 12. There are two numbers whose difference is 4; and the product of this difference, by the sum of their cubes, is 3400. What are the numbers?

Ex. 13. Several persons form a partnership, and establish a certain capital, to which each contributes ten times as many dollars as there are persons in company. They gain 6 *plus* the number of partners per cent., and the whole profit is \$392. How many partners were there?

Ex. 14. There is a number consisting of three digits such that the sum of the first and second is 9; the sum of the first and third is 12; and if the product of the three digits be increased by 38 times the first digit, the sum will be 336. Required the number.

Ans. { 636, or 725, or 814.

Ex. 15. A company of merchants have a common stock of \$4775, and each contributes to it twenty-five times as many dollars as there are partners, with which they gain as much per cent. as there are partners. Now, on dividing the profit, it is found, after each has received six times as many dollars as there are persons in the company, that there still remains \$126. Required the number of merchants.

Ans. 7, 8, or 9.

EQUATIONS OF THE FOURTH AND HIGHER DEGREES.

468. It may be easily shown that the method here employed for cubic equations is applicable to equations of every degree. For the fourth degree we shall have one more column of products, but the operations are all conducted in the same manner, as will be seen from the following example.

Ex. 1. Find the four roots of the equation

$$x^4-8x^3+14x^2+4x=8.$$

By Sturm's Theorem, we have found that these roots are all real; three positive, and one negative.

We then proceed as follows:

1	−8	+14	+ 4	=8 (5.2360679.
	−3	− 1	− 1	−5
	+2	+ 9	+44=1st divisor.	13=1st dividend.
	7	44	53.288	10.6576
	12.2	46.44	63.072=2d div'r.	2.3424=2d dividend.
	12.4	48.92	64.626747	1.93880241
	12.6	51.44	66.193068=3d div.	.40359759=3d div'd.
	12.83	51.8249	66.509117736	.399054706416
	12.86	52.2107	66.825633024=4th d.	4542883584=4th d.
	12.89	52.5974		
	12.926	52.674956		
	12.932	52.752548		

and by division we obtain the four figures 0679.

The other three roots may be found in the same manner.

Hence the four roots are . . . . $\begin{cases} -\ .7320508, \\ .7639320, \\ 2.7320508, \\ 5.2360679. \end{cases}$

Ex. 2. Find a root of the equation

$$x^5+2x^4+3x^3+4x^2+5x=20.$$

We have found, by Sturm's Theorem, that this equation has a real root between 1 and 2.

We then proceed as follows:

1+2		+3		+4		+5		+20 (1.125789.	
3		6		10		15		15	
4		10		20		35=1st divisor.		5=1st dividend.	
5		15		35		38.7171		3.87171	
6		21		37.171		42.6585=2d divisor.		1.12829=2d dividend.	
7.1		21.71		39.414		43.5027	2016	.87005	44032
7.2		22.43		41.730		44.3566	2080, 3d div'r.	.25823	55968, 3d div'd.
7.3		23.16		42.211	008	44.5731	44750625	.22286	5723753125
7.4		23.90		42.695	032	44.7902	83203125, 4th d.	3536	9873046875, 4th d.
7.5	2	24.05	04	43.182	080				
7.5	4	24.20	12	43.304	790125				
7.5	6	24.35	24	43.427	690500				
7.5	8	24.50	40						
7.6	05	24.54	2025						
7.6	10	24.58	0075						

Dividing the fourth dividend by the fourth divisor, we obtain the figures 789.

When we wish to obtain a root correct to a limited number of places, we may save much of the labor of the operation by cutting off all figures beyond a certain decimal. Thus if, in the example above, we cut off all beyond five decimal places in the successive dividends, and all beyond four decimal places in the divisors, it will not affect the first six decimal places in the root.

Ex. 3. Find the roots of the equation $x^4-12x^2+12x=3$.

$$Ans. \begin{cases} -3.907378, \\ +\ .443277, \\ +\ .606018, \\ +2.858083. \end{cases}$$

Ex. 4. Find the roots of the equation

$$x^4-16x^3+79x^2-140x=-58.$$

$$Ans. \begin{cases} +0.58579, \\ +3.35425, \\ +3.41421, \\ +8.64575. \end{cases}$$

Ex. 5. Find the roots of the equation

$$x^5-20x^4+150x^3-520x^2+806x=407.$$

$$Ans. \begin{cases} +0.934685, \\ +3.308424, \\ +3.824325, \\ +4.879508, \\ +7.053058. \end{cases}$$

Ex. 6. Required the fourth root of 18339659776.

Ans. 368.

Ex. 7. Required the fifth root of 26286674882643.

Ans. 483.

Ex. 8. There is a number consisting of four digits such that the sum of the first and second is 9; the sum of the first and third is 10; the sum of the first and fourth is 11; and if the product of the four digits be increased by 36 times the product of the first and third, the sum will be equal to 3024 diminished by 300 times the first digit. Required the number.

$$Ans. \begin{cases} 6345, \\ \text{or } 7234, \\ \text{or } 8123, \\ \text{or } 9012. \end{cases}$$

469. *Newton's Method of Approximation.*

Let $x^3+Bx^2+Cx=V$ be an equation to be solved. Find, by trial, a number, r, nearly equal to the root sought, and let $r+h$ denote the exact value of the root, so that h is a small fraction which is to be determined. Substitute $r+h$ for x in the given equation, and there will result a new equation containing only h and known quantities. Now, since h is supposed to be a small fraction, h^2 and h^3 will be small compared with h; and if we reject the terms which contain the second and third powers of h, we shall have, approximately,

$$h=\frac{r^3+Br^2+Cr-V}{-3r^2-2Br-C}.$$

This correction applied to the assumed root gives a closer approximation to the value of x. Repeat the operation with this corrected value of r, and a second correction will be obtained which will give a nearer value of the root; and, by successive repetitions, the value of the root may be obtained to any required degree of accuracy.

The value of h may, however, be found more briefly by observing that the numerator is the first member of the equation after V has been transposed and x changed to r; and the denominator is the first derived function of the numerator with a negative sign, Art. 450.

EXAMPLES.

1. Find a root of the equation $x^3+2x^2+3x=50$.

For the numerator of the value of h, we have

$$r^3+2r^2+3r-50.$$

Hence $$h=\frac{r^3+2r^2+3r-50}{-3r^2-4r-3}.$$

We find, by trial, that x is nearly equal to 3. If we substitute 3 for r, we shall have

$$h=-\frac{2}{21}.$$

Hence $x=2.9$ nearly. If we substitute this new value of r, we shall find the value of h to be $+.00228$.

Hence $x=2.90228$. If we repeat the operation with this last value of r, we shall find the value of h to be $+.0000034$.

Hence $$x=2.9022834.$$

2. Find a root of the equation $x^5-6x=10$.

Here $$h=\frac{r^5-6r-10}{-5r^4+6}.$$

Assume $r=2$, and we obtain

$$h=-\frac{5}{37}, \text{ or } -0.14.$$

Hence $x=1.86$ nearly. If we assume $r=1.86$, we shall find the value of h to be $-.021$.

Hence $x=1.839$ nearly. If we assume $r=1.839$, we shall find the value of h to be $+.00001266$.

Therefore $$x=1.83901266.$$

3. Find a root of the equation $x^3-9x=10$.

Ans. $x=3.4494897$.

4. Find a root of the equation $x^3+9x^2+4x=80$.

Ans. $x=2.4721359$.

470. *Approximation by Double Position.*—Find, by trial, two numbers, r and r', as near as possible to the true value of x; substitute them successively for x in the given equation, and let E and E′ represent the errors which result from these substitutions. We assume that the errors of the results are proportional to the errors of the assumed numbers. This supposition is not entirely correct; but if we employ numbers near to the true values, the error of this supposition is generally not very great, and the error becomes less and less the further we carry the approximation. We have then

$$E:E'::x-r:x-r'.$$

Whence, Art. 305, $E-E':r-r'::E:x-r$;

that is, *As the difference of the errors is to the difference of the two assumed numbers, so is either error to the correction required in the corresponding assumed number.*

This correction, being added to the assumed number when it is too small, or subtracted when too great, will give a near approximation to the true root. This result, and some other number, may now be used as new values of r and r' for obtaining a still nearer approximation, and so on.

It is generally most convenient to assume two numbers which differ only by unity in the last figure on the right, or one of the values of r already used, together with the approximate root, may be employed for the two assumed numbers.

This method of approximation is applicable to many equations which can not be solved by either of the preceding methods.

EXAMPLES.

1. Find one root of the equation $x^3+x^2+x-100=0$.

When 4 and 5 are substituted for x in this equation, the results are -16 and $+55$.

Hence $55+16:5-4::16:.22$.

Therefore $x=4.22$ nearly.

We now assume the two values 4.2 and 4.3, and, substituting them for x in the given equation, we obtain the results -4.072 and $+2.297$.

Hence $4.072+2.297:4.3-4.2::2.297:.036$.

Therefore $x=4.264$ nearly.

Assuming again the two values 4.264 and 4.265, and substituting them for x, we obtain the results $-.027552$ and $+.036535$.

Hence $.064087:.001::.027552:.0004299$.

Therefore $x=4.2644299$ very nearly.

2. Find one root of the equation $x^3+2x^2-23x-70=0$.

Ans. $x=5.13458$.

3. Find one root of the equation $x^4-3x^2-75x-10000=0$.

Ans. $x=10.2610$.

4. Find one root of the equation

$$x^5+3x^4+2x^3-3x^2-2x-2=0.$$

Ans. $x=1.059109$.

471. *The different Roots of Unity.*—The equation $x^n=a$ would appear to have but one root, that is, $x=\sqrt[n]{a}$; but, by Art. 436, it must have n roots; that is, *the nth root of a must have n different values.* Unity must therefore have two square roots, three cube roots, four fourth roots, five fifth roots, six sixth roots, and so on.

Ex. 1. Find the two roots of the equation $x^2=1$.

Extracting the square root, we find $x=+1$ or -1.

Ex. 2. Find the three roots of the equation $x^3=1$.

Since one root of this equation is $x=1$, the proposed equation must be divisible by $x-1$; and dividing, we obtain

$$x^2+x+1=0.$$

Now the roots of this equation are

$$x=-\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

Hence the required roots are

$$+1,\ \tfrac{1}{2}(-1+\sqrt{-3}),\text{ and }\tfrac{1}{2}(-1-\sqrt{-3}),$$

which are the cube roots of unity; and these results may be easily verified.

Ex. 3. Find the four roots of the equation $x^4=1$.

The square root of this equation is

$$x^2=+1,\text{ or }=-1.$$

Hence the required roots are

$$+1,\ -1,\ +\sqrt{-1},\ -\sqrt{-1}.$$

Ex. 4. Find the five roots of the equation $x^5=1$.

Since one root of this equation is $x=1$, the proposed equation must be divisible by $x-1$; and dividing, we obtain

$$x^4+x^3+x^2+x+1=0.$$

Dividing again by x^2, we have

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0. \qquad (1.)$$

Now put $$v=x+\frac{1}{x}, \qquad (2.)$$

whence $$v^2=x^2+2+\frac{1}{x^2},$$

which, being substituted in equation (1), gives

$$v^2+v-1=0.$$

This equation, solved by the usual method, gives

$$v=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5},\text{ or }v=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}.$$

Now equation (2) gives

$$x^2-vx=-1.$$

Whence $x=\frac{1}{2}[v+\sqrt{v^2-4}]$, and $x=\frac{1}{2}[v-\sqrt{v^2-4}]$, from which, by substituting the value of v, we obtain

$$x=\tfrac{1}{4}\left[\sqrt{5}-1\pm\sqrt{-10-2\sqrt{5}}\right],$$

and $$=\tfrac{1}{4}\left[-\sqrt{5}-1\pm\sqrt{-10+2\sqrt{5}}\right].$$

Hence the fifth roots of unity are

$$1.$$
$$\tfrac{1}{4}\left[\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}\right].$$
$$\tfrac{1}{4}\left[\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}\right].$$
$$\tfrac{1}{4}\left[-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}\right].$$
$$\tfrac{1}{4}\left[-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}\right].$$

Ex. 5. Find the six roots of the equation $x^6=1$.

These are found by taking the square roots of the cube roots. Hence we have

$$+1,\ -1,\ \tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3},\ -\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-3}.$$

Ex. 6. Find the four roots of the equation $x^4=-1$, or $x^4+1=0$.

The first member may be made a complete square by adding $2x^2$; that is, $$x^4+2x^2+1=2x^2,$$

whence $$x^2+1=\pm x\sqrt{2}.$$

By transposition and completing the square,

$$x^2\pm x\sqrt{2}+\tfrac{1}{2}=-\tfrac{1}{2}.$$

Hence $$x\pm\tfrac{1}{2}\sqrt{2}=\pm\tfrac{1}{2}\sqrt{-2};$$

that is, $$x=\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2},$$

or $$-\tfrac{1}{2}\sqrt{2}\pm\tfrac{1}{2}\sqrt{-2}.$$

These four values, together with the four values found in Ex. 3, are the eight roots of the equation

$$x^8=1.$$

EXAMPLES FOR PRACTICE.

EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY.

Ex. 1. Given $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}$, to find the value of x. *Ans.* $x=139\frac{1}{2}$.

Ex. 2. Given $a(2x+19b-10a)=b(x+7b)$, to find the value of x. *Ans.* $x=5a-7b$.

Ex. 3. Given $2-\frac{5+x}{7}=1-\frac{9-x}{14}$, to find the value of x.

Ans. $x=4\frac{1}{3}$.

Ex. 4. Given $m+\frac{x}{a}=n-p-\frac{x}{b}$, to find the value of x.

Ans. $x=\frac{(n-p-m)ab}{a+b}$.

Ex. 5. Given $\frac{2x-3}{15}-\frac{4x-9}{20}=\frac{8x-27}{30}-\frac{16x-81}{24}-\frac{9}{40}$, to find the value of x. *Ans.* $x=6$.

Ex. 6. Given $a^3+a^2b+ab^2+b^3=\frac{a^4-b^4}{x}$, to find the value of x.

Ans. $x=a-b$.

Ex. 7. Given $\frac{a^4-b^4}{a^2(a-b)}-\frac{a^2x+b^3}{a^2}=2b+\frac{b^2}{a}$, to find the value of x. *Ans.* $x=a-b$.

Ex. 8. Given $\frac{3x}{5}-\frac{7x}{10}+\frac{3x}{4}-\frac{7x}{8}=-15$, to find the value of x.

Ans. $x=66\frac{2}{3}$.

Ex. 9. Given $11\frac{1}{2}x=\frac{11}{8}x+66\frac{7}{8}-5x-9\frac{1}{4}$, to find the value of x. *Ans.* $x=3\frac{98}{121}$.

Ex. 10. Given $\frac{3a+x}{x}-5=\frac{6}{x}$, to find the value of x.

Ans. $x=\frac{3a-6}{4}$.

Ex. 11. Given $c=a+\frac{m(a-x)}{3a+x}$, to find the value of x.

Ans. $x=\frac{a(m-3c+3a)}{c-a+m}$.

Ex. 12. Given $\frac{1-2x}{3}-\frac{4-5x}{6}=-\frac{13}{42}$, to find the value of x.

Ans. $x=\frac{1}{7}$.

Ex. 13. Given $(m-x)(n-x)=(p+x)(x-q)$, to find the value of x.

Ans. $x=\frac{mn+pq}{m+n+p-q}$.

Ex. 14. Given $8x-28=(4x+21)\frac{6x-22}{3x+14}$, to find the value of x.

Ans. $x=7$.

Ex. 15. Given $x=a+\frac{bc}{d}+\frac{cfx}{de}$, to find the value of x.

Ans. $x=\frac{(ad+bc)e}{de-cf}$.

Ex. 16. Given $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$, to find the value of x.

Ans. $x=\frac{ac(1-3ab)}{c-ad}$.

Ex. 17. Given $(8-3x)^2+(4-4x)^2=(9-5x)^2$, to find the value of x.

Ans. $x=\frac{1}{10}$.

Ex. 18. Given $\frac{16x+7}{24}+\frac{x-16}{177-9x}=\frac{2x+1}{3}$, to find the value of x.

Ans. $x=17$.

Ex. 19. Given $\frac{9x+10}{11x-12}-\frac{8+5x}{40}=\frac{12}{13}-\frac{1}{8}x$, to find the value of x.

Ans. $x=7$.

Ex. 20. Given $\frac{1-2x}{3-4x}-\frac{5-6x}{7-8x}=\frac{8}{3}\cdot\frac{1-3x^2}{21-52x+32x^2}$, to find the value of x.

Ans. $x=\frac{2}{3}$.

PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE WITH ONE UNKNOWN QUANTITY.

Prob. 1. Said an old miser, For 50 years I have saved 200 dollars annually; and for many years, each of my four sons has saved annually the same sum, viz., the oldest for 27 years past, the second since 24 years, the third since 19, and the fourth since 16 years. How long since the savings of the four sons amounted in the aggregate to as much as those of the father?

Ans. 12 years.

Prob. 2. From four towns, A, B, C, D, lying along the same road, four persons start in the stage-coach for the same place, E. The distance from A to B is 19 miles, from B to C 3 miles, and from C to D 5 miles. It subsequently appeared that the person who started from A paid as much fare as the three other persons together; and the fare per mile was the same for each. It is required to determine the distance from D to E.

Ans. 7 miles.

Prob. 3. Five towns, A, B, C, D, E, are situated along the same highway. The distance from A to B is 37 miles, from B to D 34, and from D to E 14 miles. A merchant at C, situated between A and D, receives at one time 8 tons of goods from A, and 6 tons from B. At another time he receives 11 tons from D, and 9 from E, and in the latter case he paid the same amount for freight as in the former, the rate of transportation being the same in both cases. It is required to compute the distance from B to C. *Ans.* 15 miles.

Prob. 4. If 20 quarts of water flow into a reservoir every 3 minutes, after a certain time it will still lack 40 quarts of being full. But if 52 quarts flow into it every 5 minutes during the same period, 72 quarts of water will have overflown. What is the capacity of the reservoir, and how many quarts of water must flow into it every minute in order that it may be just filled in the time before mentioned?

Ans. The capacity of the reservoir is 240 quarts, and 8 quarts must flow into it every minute.

Prob. 5. A mason, by working 10 hours daily, could complete in a week as much over 888 cubic feet of wall as at present he completes *less* than 888 cubic feet, working only $8\frac{1}{2}$ hours daily. How many cubic feet of wall does he now complete weekly? *Ans.* 816 cubic feet.

Prob. 6. After a certain time I have \$670 to pay, and $4\frac{1}{2}$ months later I have \$980 to pay. I settle both bills at once, at $4\frac{4}{5}$ per cent. discount, for \$1594.41. When did the first sum become due? *Ans.* After $5\frac{3}{4}$ months.

Prob. 7. A merchant gains 8 per cent. when he sells a hogshead of oil at 36 dollars. How much per cent. does he gain or lose when he sells a hogshead at 32 dollars?

Ans. He loses 4 per cent.

Prob. 8. A merchant loses $2\frac{1}{2}$ per cent. when he sells a bag of coffee for 39 dollars. How much per cent. does he gain or lose when he sells a bag of coffee for $41\frac{1}{2}$ dollars?

Ans. He gains $3\frac{3}{4}$ per cent.

Prob. 9. A merchant owes \$2007, to be paid after 5 months, \$3395 after 7 months, and \$6740 after 13 months. When should the entire sum of \$12,142 be paid, so that neither party may sustain any loss? *Ans.* After 10 months.

Prob. 10. A merchant has three sums of money to pay, viz., \$1013 after $3\frac{1}{2}$ months, \$431 four months later, and the third sum still four months later. How large is the third sum, supposing he could pay the three bills together in $6\frac{1}{4}$ months without loss or gain? *Ans.* \$428.

Prob. 11. A merchant has two kinds of tobacco; the one cost 40 cents per pound, the other 24 cents. He wishes to mix the two kinds together, so that he may sell it at 34 cents per pound without loss or gain. How much must he take of each sort in order to have 64 pounds of the mixture?

Ans. 40 pounds of the better sort, and 24 pounds of the poorer.

Prob. 12. A vinegar dealer wishes to dilute his vinegar with water. At present he sells his vinegar at 6 dollars per hogshead (120 quarts). How much water must he add to $29\frac{1}{2}$ hogsheads in order to be able to sell the mixture at 4 cents per quart? *Ans.* $7\frac{3}{8}$ hogsheads.

Prob. 13. A metallic compound consists of 4 parts copper and 3 parts silver. How much copper must be added to $94\frac{1}{2}$ pounds of the compound, in order that the proportions may be 7 parts of copper to 2 parts of silver? *Ans.* $87\frac{3}{4}$ pounds.

Prob. 14. In 255 pounds of spirit of wine, water and pure alcohol are combined by weight in the ratio of 2 to 3. How much water must be extracted by distillation, in order that the ratio of the water to the alcohol may be 3 to 17 by weight?

Ans. 75 pounds.

Prob. 15. It is required to diminish each of the factors of the two unequal products, 52×45 and 66×37, by the same number, so that the new products may be equal to each other. What is that number? *Ans.* 17.

Prob. 16. The square of a certain number is 1188 greater than the square of a number smaller by 6 than the former. What is that number? *Ans.* 102.

Prob. 17. I have a certain number of dollars in my possession, which I undertook to arrange in the form of a square, and found that I wanted 25 dollars to complete the square; but if I diminish each side of the square by 2, there remain 31 dollars over. How many dollars have I? *Ans.* 200.

Prob. 18. A vine-tiller has a rectangular garden, whose length is to its breadth as 7 to 5, which he wishes to plant with vines. If he sets the plants at a certain uniform distance from each other, he finds that he has 2832 plants remaining. But if he places them nearer together, so as to make 14 more on each longer side, and 10 more on each shorter side, he has only 172 plants remaining. How many plants has he?

Ans. 14,172.

Prob. 19. In the composition of a certain quantity of gunpowder, the nitre was ten pounds more than two thirds of the whole; the sulphur was four and a half pounds less than one sixth of the whole; and the charcoal was two pounds less than one seventh of the nitre. How many pounds of gunpowder were there? *Ans.* 69 pounds.

Prob. 20. There are three numbers in the ratio of 3, 4, and 5. Five times the first number, together with four times the second number, and three times the third number, make 690. What are the three numbers?

Ans. 45, 60, and 75.

Prob. 21. Divide the number 165 into five such parts that the first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by five, may all be equal.

Ans. 19, 18, 23, 5, and 100.

Prob. 22. A criminal, having escaped from prison, traveled ten hours before his escape was known. He was then pursued, so as to be gained upon three miles an hour. After his pursuers had traveled eight hours, they met an express going at the same rate as themselves, who met the criminal two hours and twenty-four minutes before. In what time from the commencement of the pursuit will they overtake him?

Ans. 20 hours.

Prob. 23. There is a wagon with a mechanical contrivance by which the difference of the number of revolutions of the wheels on a journey is noted. The circumference of the fore wheel is a feet, and of the hind wheel b feet. What is the distance gone over when the fore wheel has made n revolutions more than the hind wheel?

Ans. $\frac{abn}{b-a}$ feet.

Prob. 24. A cistern can be filled by four pipes; by the first in a hours, by the second in b hours, by the third in c hours, and by the fourth in d hours. In what time will the cistern be filled when the four pipes are opened at once?

Ans. $\frac{abcd}{abc+abd+acd+bcd}$ hours.

EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

Ex. 1. $\left\{\begin{aligned} \frac{x}{5}+\frac{y}{6}&=18 \\ \frac{x}{2}-\frac{y}{4}&=21 \end{aligned}\right\}$ *Ans.* $\left\{\begin{aligned} x&=60. \\ y&=36. \end{aligned}\right.$

Ex. 2. $\begin{cases} \dfrac{x+y}{2}-\dfrac{x-y}{3}=8 \\ \dfrac{x+y}{3}+\dfrac{x-y}{4}=11 \end{cases}$ *Ans.* $\begin{cases} x=18. \\ y=\ \ 6. \end{cases}$

Ex. 3. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=1 \\ \dfrac{x}{3}+\dfrac{y}{4}=1 \end{cases}$ *Ans.* $\begin{cases} x=-6. \\ y=12. \end{cases}$

Ex. 4. $\begin{cases} x+\dfrac{y}{11}=71 \\ y-\dfrac{x}{13}=61 \end{cases}$ *Ans.* $\begin{cases} x=65. \\ y=66. \end{cases}$

Ex. 5. $\begin{cases} \dfrac{2x+7y}{5}-1=\frac{2}{3}(2x-6y+1 \\ x=4y \end{cases}$ *Ans.* $\begin{cases} x=4. \\ y=1. \end{cases}$

Ex. 6. $\begin{cases} \dfrac{11x-5y}{22}=\dfrac{3x+y}{32} \\ 8x-5y=1 \end{cases}$ *Ans.* $\begin{cases} x=7. \\ y=11. \end{cases}$

Ex. 7. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=1 \\ \dfrac{x}{3a}+\dfrac{y}{6b}=\frac{2}{3} \end{cases}$ *Ans.* $\begin{cases} x=3a. \\ y=-2b. \end{cases}$

Ex. 8. $\begin{cases} \dfrac{x}{a+b}-\dfrac{y}{a-b}=\dfrac{1}{a+b} \\ \dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a-b} \end{cases}$ *Ans.* $\begin{cases} x=\dfrac{a}{a-b}. \\ y=\dfrac{b}{a+b}. \end{cases}$

Ex. 9. $\begin{cases} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5} \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3} \end{cases}$ *Ans.* $\begin{cases} x=12. \\ y=6. \end{cases}$

Ex. 10. $\begin{cases} \dfrac{7+8x}{10}-\dfrac{3x-6y}{2x-8}=4-\dfrac{9-4x}{5} \\ \dfrac{6y+9}{4}=3\frac{1}{4}+\dfrac{3y+4}{2}-\dfrac{3y+5x}{4y-6} \end{cases}$ *Ans.* $\begin{cases} x=9. \\ y=7. \end{cases}$

Ex. 11. $\left\{\begin{array}{l}\frac{x}{a}+\frac{y}{b}=4\\ \frac{x}{a}+\frac{z}{c}=4\\ \frac{y}{b}+\frac{z}{c}=4\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=2a.\\ y=2b.\\ z=2c.\end{array}\right.$

Ex. 12. $\left\{\begin{array}{l}5x-6y+4z=15\\ 7x+4y-3z=19\\ 2x+y+6z=46\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=3.\\ y=4.\\ z=6.\end{array}\right.$

Ex. 13. $\left\{\begin{array}{l}x=21-4y\\ z=9-\frac{2x}{3}\\ y=64-7\frac{1}{2}z\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=1\frac{2}{3}.\\ y=4\frac{5}{6}.\\ z=7\frac{8}{9}.\end{array}\right.$

Ex. 14. $\left\{\begin{array}{l}x+y+z=5\\ 3x-5y+7z=75\\ 9x-11z+10=0\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=5.\\ y=-5.\\ z=5.\end{array}\right.$

Ex. 15. $\left\{\begin{array}{l}3x-5y+4z=5\\ 7x+2y-3z=2\\ 4x+3y-z=7\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=1.\\ y=2.\\ z=3.\end{array}\right.$

Ex. 16. $\left\{\begin{array}{l}x-2y+3z=6\\ 2x+3y-4z=20\\ 3x-2y+5z=26\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=8.\\ y=4.\\ z=2.\end{array}\right.$

Ex. 17. $\left\{\begin{array}{l}7x-3y=1\\ 4z-7y=1\\ 11z-7u=1\\ 19x-3u=1\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=4.\\ y=9.\\ z=16.\\ u=25.\end{array}\right.$

Ex. 18. $\left\{\begin{array}{l}3u-2y=2\\ 2x+3y=39\\ 5x-7z=11\\ 4y+3z=41\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=12\\ y=5.\\ z=7.\\ u=4.\end{array}\right.$

Ex. 19. $\left\{\begin{array}{l}2x-3y+2z=13\\ 4y+2z=14\\ 4u-2x=30\\ 5y+3u=32\end{array}\right\}$ *Ans.* $\left\{\begin{array}{l}x=3.\\ y=1.\\ z=5.\\ u=9.\end{array}\right.$

Ex. 20. $\begin{cases} 1\frac{2}{3}x+2\frac{3}{4}y \ =105 \\ 3\frac{4}{5}x+4\frac{5}{6}z \ =317 \\ 5\frac{6}{7}z+6\frac{7}{8}u \ =741 \\ 7\frac{8}{9}u+8\frac{9}{10}x=835 \end{cases}$ *Ans.* $\begin{cases} x=30. \\ y=20. \\ z=42. \\ u=72. \end{cases}$

Ex. 21. $\begin{cases} 3x-4y+3z+3v-6u=11 \\ 3x-5y+2z-4u=11 \\ 10y-3z+3u-2v=2 \\ 5z+4u+2v-2x=3 \\ 6u-3v+4x-2y=6 \end{cases}$ *Ans.* $\begin{cases} x=\ \ 2. \\ y=\ \ 1. \\ z=\ \ 3. \\ u=-1. \\ v=-2. \end{cases}$

PROBLEMS INVOLVING EQUATIONS OF THE FIRST DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

Prob. 1. Two sums of money, which were put out at interest, the one at 5 per cent., the other at $4\frac{1}{2}$ per cent., yielded in one year $284.40 interest. If the former sum had been put out at $4\frac{1}{2}$ per cent., and the latter at 5 per cent., they would have yielded $4.50 less interest. What were the two sums of money?
Ans. One was $3420, the other $2520.

Prob. 2. There is a number consisting of two digits; the number is equal to three times the sum of its digits, and if it be multiplied by three, the result will be equal to the square of the sum of its digits. Find the number. *Ans.* 27.

Prob. 3. A merchant sold two bales of goods for the sum of $\$987\frac{5}{8}$, the first at a loss of $8\frac{3}{4}$ per cent., the second at a loss of $11\frac{1}{4}$ per cent. If he had sold the first at a loss of $11\frac{1}{4}$ per cent., and the second at a loss of $8\frac{3}{4}$ per cent., he would have received the sum of $\$992\frac{3}{8}$. How much did each bale cost?
Ans. The first $455, the second $645.

Prob. 4. Two messengers, A and B, from two towns distant $57\frac{1}{2}$ miles from each other, set out to meet each other. If A starts $5\frac{3}{4}$ hours earlier than B, they will meet in $6\frac{1}{8}$ hours after B starts; but if B starts $5\frac{3}{4}$ hours earlier than A, they will meet in $5\frac{5}{8}$ hours after A starts. How many miles does each travel in an hour? *Ans.* A 3 miles, and B $3\frac{4}{7}$ miles.

Prob. 5. A jeweler has two masses of gold of different degrees of fineness. If he mixes 10 ounces of the one with 5 ounces of the other, he obtains gold 11 carats fine; but if he mixes $7\frac{1}{2}$ ounces of the former with $1\frac{1}{2}$ ounces of the latter, he obtains a mixture 10 carats fine. What was the fineness of each mass? *Ans.* The one 9 carats, the other 15 carats.

Prob. 6. A farmer has a certain number of oxen, and provender for a certain number of days. If he sells 75 oxen, his provender will last 20 days longer; but if he buys 100 more oxen, his provender will be exhausted 15 days sooner. How many oxen has he, and how many days will the provender last?

Ans. 300 oxen, and the provender will last 60 days.

Prob. 7. A certain number of laborers remove a pile of stones in 6 hours from one place to another. If there had been 2 more laborers, and if each laborer had each time carried 4 pounds more, the pile would have been removed in 5 hours; but if there had been 3 less laborers, and if each laborer had each time carried 5 pounds less, it would have required 8 hours to remove the pile. How many laborers were there, and how much did each carry at one time?

Ans. There were 18 laborers, and each carried 50 pounds.

Prob. 8. A heavy wagon requires a certain time to travel from A to B. A second wagon, which every 4 hours travels 5 miles less than the first, requires 4 hours more than the first to go from A to B. A third wagon, which every 3 hours travels $8\frac{3}{4}$ miles more than the second, requires 7 hours less than the second to make the same journey. How far is A from B, and what time does each wagon require to travel this distance?

Ans. From A to B is 60 miles; the first wagon requires 12 hours, the second 16, and the third 9 hours.

Prob. 9. I have two equal sums to pay, one after 9, and the other after 15 months. If I settle them both at once, at the same rate of discount, I must pay for the first sum $1208, and for the second $1160. How much was each sum, and at what per cent. was the discount reckoned?

Ans. $1280, and the discount was $7\frac{1}{2}$ per cent.

Prob. 10. A small square lies with one angle in the angle of a larger square. The excess of the side of the larger square above that of the smaller is 118 feet; the excess of the square itself is 26,432 square feet. What are the contents of each of the two squares?

Ans. The one 29,241, the other 2809 square feet.

Prob. 11. It is required to find two numbers whose sum, difference, and product are in the ratio of the numbers 5, 1, and 18.

Ans. 9 and 6.

Prob. 12. Two numbers are in the ratio of 7 to 3, and their difference is to their product as 1 to 21. What are the numbers?

Ans. 28 and 12.

Prob. 13. Three towns, A, B, and C, lie at the angles of a triangle. From A by B to C is 164 miles; from B by C to A is 194 miles; and from C by A to B is 178 miles. How far are A, B, and C from each other?

Ans. From A to B 74 miles, from B to C 90, and from C to A 104 miles.

Prob. 14. A railway train, after traveling for one hour, meets with an accident which delays it one hour, after which it proceeds at three fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour and 20 minutes sooner. Required the length of the line.

Ans. 140 miles; original rate 20 miles per hour.

Prob. 15. A railway train, running from New York to Albany, meets on the way with an accident, which causes it to diminish its speed to $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b miles nearer Albany, the train would have been c hours late. Find the rate of the train before the accident occurred.

Ans. $\frac{b(n-1)}{a-c}$ miles per hour.

Prob. 16. Three boys are playing with marbles. Said A to B, Give me 5 marbles, and I shall have twice as many as you will have left. Said B to C, Give me 13 marbles, and I shall have three times as many as you will have left. Said C to A, Give me 3 marbles, and I shall have six times as many as you will have left. How many marbles had each boy?

Ans. A had 7, B 11, and C 21 marbles.

Prob. 17. It is required to divide the number 232 into three parts such that, if to the first we add half the sum of the other two, to the second we add one third the sum of the other two, and to the third we add one fourth the sum of the other two, the three results thus obtained shall be equal. What are the parts?

Ans. The first 40, the second 88, and the third 104.

Prob. 18. Four towns, A, B, C, and D, are situated at the angles of a quadrilateral figure. When I travel from A by B and C to D, I pay $6.10 passage-money; when I travel from A by D and C to B, I pay $5.50. From A by B to C, I pay the same as from A by D to C; but from B by A to D, I pay 40 cents less than from B by C to D. What are the distances of the four towns from each other, supposing I paid in each case 10 cents per mile?

Ans. From A to B 21, from B to C 17, from C to D 23, and from D to A 15 miles.

Prob. 19. Four players, A, B, C, and D, play four games at cards. At the first game A, B, and C win, and each of them doubles his money; at the second game A, B, and D win, each of them doubling the money he had at the commencement of that game; at the third game A, C, and D win; and at the fourth game B, C, and D win; and at each game each winner won as much money as he had at the commencement of that game. They now count their money, and find that each has $64. How much had each before commencing play?

Ans. A had $20, B had $36, C had $68, and D had $132.

Prob. 20. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half an hour before B, but, missing his way, goes a mile and back again needlessly, during which he walks at twice his former pace, and reaches the top six minutes before B. C starts twenty minutes after A and B, and, walking at the rate of two and one seventh miles per hour, arrives at the summit ten minutes after B. Find the rates of walking of A and B, and the distance from the foot to the summit of the mountain.

Ans. $2\frac{1}{2}$, 2; distance 5 miles.

Prob. 21. Find three numbers such that if six be subtracted from the first and second, the remainders will be in the ratio of 2 : 3; if thirty be added to the first and third, the sums will be in the ratio of 3 : 4; but if ten be subtracted from the second and third, the remainders will be as 4 : 5.

Ans. 30, 42, 50.

Prob. 22. A and B engage to reap a field of wheat in twelve days. The times in which they could severally reap an acre are as 2 : 3. After some days, finding themselves unable to finish it in the stipulated time, they call in C to help them, whose rate of working was such that, if he had wrought with them from the beginning, it would have been finished in nine days. Also, the times in which he could have reaped the field with A alone, and with B alone, are in the ratio of 7 : 8. When was C called in? *Ans.* After six days.

EQUATIONS OF THE SECOND DEGREE WITH ONE UNKNOWN QUANTITY.

A.—INCOMPLETE EQUATIONS OF THE SECOND DEGREE.

Ex. 1. Given $\frac{x+18}{x+2}+\frac{x-18}{x-2}=\frac{5}{3}$, to find the values of x.

Ans. $x=\pm 14$.

Ex. 2. Given $\sqrt{\frac{5}{x^2}+49}-\sqrt{\frac{5}{x^2}-49}=7$, to find the values of x. *Ans.* $x=\pm\frac{2}{7}$.

Ex. 3. Given $\frac{x}{5}+\frac{5}{x}=\frac{x}{2}+\frac{2}{x}$, to find the values of x.

Ans. $x=\pm\sqrt{10}$.

Ex. 4. Given $x+\sqrt{a+x^2}=\frac{a^2+a}{2\sqrt{a+x^2}}$, to find the values of x.

Ans. $x=\pm\frac{1}{2}(a-1)$.

Ex. 5. Given $\sqrt{\frac{3m^2}{x^2}+m^2-3}=m+1-\sqrt{\frac{3m^2}{x^2}-2}$, to find the values of x. *Ans.* $x=\pm m$.

Ex. 6. Given $\sqrt{\frac{560}{x^2}+29}-\sqrt{\frac{560}{x^2}-34}=7$, to find the values of x. *Ans.* $x=\pm 4$.

Ex. 7. Given $\frac{1}{1-\sqrt{1-x^2}}-\frac{1}{1+\sqrt{1-x^2}}=\frac{\sqrt{3}}{x^2}$, to find the values of x. *Ans.* $x=\pm\frac{1}{2}$.

Ex. 8. Given $27(7-x)^2-43=77-3(7-x)^2$, to find the values of x.

Remark. Put $7-x=y$; first find the value of y, and thence the value of x.

Ans. $x=5$ or 9.

Ex. 9. Given $\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$, to find the values of x.

Ans. $x=\pm\frac{2a\sqrt{b}}{1+b}$.

Ex. 10. Given $\frac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\frac{ab^2}{x-a}$, to find the values of x.

Ans. $x=\frac{a(1\pm b)^2}{1\pm 2b}$.

Ex. 11. Given $\frac{\sqrt{a+x}}{\sqrt{x}}+\frac{\sqrt{a-x}}{\sqrt{x}}=\sqrt{\frac{x}{b}}$, to find the values of x.

Ans. $x=\pm 2\sqrt{ab-b^2}$.

Ex. 12. Given $\frac{\sqrt{1+x}}{1+\sqrt{1+x}}=\frac{\sqrt{1-x}}{1-\sqrt{1-x}}$, to find the values of x.

Ans. $x=\pm\frac{1}{2}\sqrt{3}$.

B.—COMPLETE EQUATIONS OF THE SECOND DEGREE.

Ex. 13. Given $557x=5801\frac{1}{4}+8x^2$, to find the values of x.
Ans. $x=56\frac{7}{8}$ or $12\frac{3}{4}$.

Ex. 14. Given $(7x)^2-7x=1$, to find the values of x.
Ans. $x=0.2311477$ or -0.0882905.

Ex. 15. Given $12x^2=21+\frac{1}{4}x$, to find the values of x.
Ans. $x=1\frac{1}{3}$ or $-1\frac{5}{16}$.

Ex. 16. Given $57x-18x^2+145=0$, to find the values of x.
Ans. $x=4\frac{5}{6}$ or $-1\frac{2}{3}$.

Ex. 17. Given $\frac{5x}{21}(x+1)-\frac{1}{7}(2x^2+x-1)=\frac{4}{35}(x+1)$, to find the values of x. *Ans.* $x=-1$ or $\frac{3}{5}$.

Ex. 18. Given $\frac{x-6}{x-12}-\frac{x-12}{x-6}=\frac{5}{6}$, to find the values of x.

Ans. $x=24$ or $\frac{42}{5}$.

Ex. 19. Given $\frac{x+4}{x-4}+\frac{x-4}{x+4}=\frac{10}{3}$, to find the values of x.

Ans. $x=8$ or -8.

Ex. 20. Given $\frac{4}{x+1}+\frac{5}{x+2}=\frac{12}{x+3}$, to find the values of x.

Ans. $x=3$ or $-\frac{5}{3}$.

Ex. 21. Given $\frac{2+3x}{1-4x}-\frac{6-5x}{7x-25}=\frac{16-x}{28x-193}$, to find the values of x.

Ans. $x=8$ or $-2\frac{111}{178}$.

Ex. 22. Given $\frac{2x-3}{3x-5}+\frac{3x-5}{2x-3}=\frac{5}{2}$, to find the values of x.

Ans. $x=\frac{7}{4}$ or 1.

Ex. 23. Given $\frac{x+1}{x-1}+\frac{x+2}{x-1}=\frac{2(x+3)}{x-3}$, to find the values of x.

Ans. $x=1$ or $-\frac{3}{7}$.

Ex. 24. Given $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$, to find the values of x.

Ans. $x=2+\sqrt{3}$, or $-2(2+\sqrt{3})$.

Ex. 25. Given $\frac{\sqrt{x}}{21-\sqrt{x}}+\frac{21-\sqrt{x}}{\sqrt{x}}=2\frac{1}{2}$, to find the values of x.

Ans. $x=49$ or 196.

Ex. 26. Given $\sqrt[4]{x}+\sqrt{x}=20$, to find the values of x.

Ans. $x=(+4)^4=256$ or $(-5)^4=625$.

Ex. 27. Given $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$, to find the values of x.

Ans. $x=-a$ or $-b$.

Ex. 28. Given $\frac{a-x}{b+x}-\frac{b-x}{a+x}=\frac{a+b}{a-b}$, to find the values of x.

Ans. $x=\frac{\pm\sqrt{5}(a-b)-(a+b)}{2}$.

Ex. 29. Given $\frac{a+x+\sqrt{2ax+x^2}}{a+x}=b$, to find the values of x.

Ans. $x=\frac{\pm a(1\mp\sqrt{2b-b^2})}{\sqrt{2b-b^2}}$.

Ex. 30. Given $x^4-4x^3+7x^2-6x=18$, to find the values of x by a quadratic equation. *Ans.* $x=3$ or -1, or $1\pm\sqrt{-5}$.

PROBLEMS INVOLVING EQUATIONS OF THE SECOND DEGREE WITH ONE UNKNOWN QUANTITY.

Prob. 1. It is required to find three numbers which are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, and the sum of whose squares is 10,309.

Ans. 78, 52, 39.

Prob. 2. A gentleman buys a certain number of pounds of salt, four times as much sugar, and eight times as much coffee, and for each pound of the three articles he paid as many cents as he bought pounds of that article. For the whole he paid $3.24. How many pounds of coffee did he buy?

Ans. 16 pounds.

Prob. 3. A rectangular garden was 37 feet broad and 259 feet long. Its breadth was increased by a certain number of feet, and its length diminished by seven times that number, by which means its area was diminished 63 square feet. By how many feet was the breadth increased? *Ans.* 3 feet.

Prob. 4. Find that number whose square added to its cube is nine times the next higher number. *Ans.* 3.

Prob. 5. A sets out from New York to Chicago, and B at the same time from Chicago to New York, and they travel uniformly; A reaches Chicago 16 hours, and B reaches New York 36 hours after they have met on the road. Find in what time each has performed the journey.

Ans. A 40 hours, B 60 hours.

Prob. 6. A square vineyard, in which the vines are set in squares so as to be uniformly four feet apart, is to be replanted so that the vines may be uniformly $3\frac{1}{2}$ feet apart. Supposing 8640 more vines are required for this change, what must be the length of each side of the vineyard? *Ans.* 672 feet.

Prob. 7. A glass mirror, 33 inches high and 22 inches wide, is to be set in a frame of uniform breadth, such that the surface of the frame shall be just equal to that of the glass. What must be the breadth of the frame? *Ans.* $5\frac{1}{2}$ inches.

Prob. 8. Required the solution of the preceding problem, if we represent the height of the mirror by a and its breadth by b, and it is required that the surface of the frame shall be p times that of the mirror.

Ans. $\frac{\sqrt{(a+b)^2+4abp}-(a+b)}{4}$.

Prob. 9. Sixty pounds of a certain quality of sugar cost $2.40 less than sixty pounds of another quality. If I buy sugar of each quality to the amount of $5.04, I obtain of the first kind 8 pounds more than of the second. What was the price of a pound of each kind?

Ans. One 14 cents, the other 18 cents.

Prob. 10. A gentleman bought a horse for a certain sum. He afterward sold him for $144, and gained as much per cent. as the horse cost him. How much did he pay for the horse?

Ans. 80 dollars.

Prob. 11. A merchant buys a certain number of barrels of flour for $216. At another time he expended the same sum of money for flour, but obtained three barrels less, the price of flour having risen one dollar per barrel. How many barrels did he buy in the first case? *Ans.* 27 barrels.

Prob. 12. A and B contribute together $3400 in trade, A for 12 and B for 16 months. In the distribution, A received $2070, capital and profits, and B received $1920. What was each one's capital? *Ans.* A contributed $1800, and B $1600.

Prob. 13. Supposing the mass of the earth to be 80 times that of the moon, their distance 240,000 miles, and the force of attraction to vary directly as the quantity of matter, and inversely as the square of the distance, at what point between them will a third body be equally attracted by the earth and moon?

Ans. 24,134 miles from the moon.

Prob. 14. A wall was completed in $5\frac{1}{2}$ days by two masons, one of whom commenced work $1\frac{1}{2}$ days later than the other. In order to complete the wall alone, the first would have required 3 days less than the second. In how many days could each alone complete the wall?

Ans. The first in 8, the second in 11 days.

Prob. 15. A courier goes from a place, A, to a place, B, in

14 hours. At the same time, another courier starts for B from a place 10 miles further distant, and expects to reach B at the same time with the first, by gaining half an hour in every 20 miles. What is the distance from A to B? *Ans.* 70 miles.

Prob. 16. From two towns, A and B, which are 104 miles distant from each other, two wagons start at the same time, and meet after $10\frac{1}{2}$ hours. One requires for every 8 miles a quarter of an hour more than the other. How much time does each require to travel one mile?

Ans. The one $\frac{7}{32}$, the other $\frac{3}{16}$ of an hour.

Prob. 17. Two messengers start at the same time from two towns, A and B, the first toward B, the other toward A, and, upon meeting, it appeared that the first had traveled 12 miles more than the second; also, that if each should continue on at his former rate, the first would arrive at B in 9 hours, and the latter at A in 16 hours. What is the distance from A to B?

Ans. 84 miles.

Prob. 18. Two messengers start from the two towns, A and B, to travel toward each other, but one started two hours earlier than the other. They meet each other $2\frac{1}{12}$ hours after the starting of the second messenger, and they reach the towns A and B at the same instant. In how many hours did each messenger perform the journey?

Ans. The one in 7, the other in 5 hours.

Prob. 19. Two travelers start from two towns, A and B, whose distance from each other is 910 miles, and travel uniformly toward each other. If the first starts 56 hours before the second, they will meet half way between A and B. If both start at the same instant, at the end of 20 hours they will still be 550 miles from each other. How many hours does each traveler require to accomplish the distance from A to B?

Ans. One 182 hours, the other 70 hours.

Prob. 20. A grocer has a cask containing 20 gallons of brandy, from which he draws off a certain quantity into another cask of equal size, and, having filled the last with water, the first cask was filled with the mixture. It now appears that if $6\frac{2}{3}$ gallons of the mixture are drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

Ans. 10 gallons.

Prob. 21. Two merchants sold the same kind of cloth. The second sold three yards more of it than the first, and together they received \$35. The first said to the second, I should have received \$24 for your cloth; the other replied, I should have received $\$12\frac{1}{2}$ for yours. How many yards did each of them sell?

Ans. The first merchant 5 or 15 yards, the second merchant 8 or 18 yards.

Prob. 22. A and B traveled on the same road, and at the same rate, from Cumberland to Baltimore. At the 50th milestone from Baltimore A overtook a drove of geese, which were proceeding at the rate of three miles in two hours, and two hours afterward met a wagon, which was moving at the rate of nine miles in four hours. B overtook the same drove of geese at the 45th milestone, and met the same wagon 40 minutes before he came to the 31st milestone. Where was B when A reached Baltimore? *Ans.* 25 miles from Baltimore.

EQUATIONS OF THE SECOND DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

Ex. 1. Given $\left.\begin{array}{l}(13x)^2+2y^2=177,\\(2y)^2-13x^2=\ \ 3,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm1$, $y=\pm2$.

Ex. 2. Given $\left.\begin{array}{l}x^2+y^2:x^2-y^2::25:7,\\xy=48,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm8$, $y=\pm6$.

Ex. 3. Given $\left.\begin{array}{l}2(x+4)^2-\ 5(y-7)^2=\ \ \ 75,\\7(x+4)^2+15(y-7)^2=1075,\end{array}\right\}$ to find the values of x and y.

Remark. Put $x+4=z$, and $y-7=v$. First determine z and v, and thence x and y.

Ans. $x=+6$ or -14, $y=12$ or 2.

Ex. 4. Given $\left.\begin{array}{l}(x+y)^2-2x^2\ =\ 49,\\3x^2+4(x+y)^2=372,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm4$, $y=\pm5$, or ±13.

Ex. 5. Given $\left.\begin{array}{l} 2x+3y=37, \\ \frac{1}{x}+\frac{1}{y}=\frac{14}{45}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=5$ or $\frac{333}{28}$, $y=9$ or $\frac{370}{84}$.

Ex. 6. Given $\left.\begin{array}{l} \frac{x^2}{y}+\frac{y^2}{x}=9, \\ x+y=6, \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$ or 2, $y=2$ or 4.

Ex. 7. Given $\left.\begin{array}{l} x^2+y^2=10000, \\ x+y=124, \end{array}\right\}$ to find the values of x and y.

Ans. $x=96$ or 28, $y=28$ or 96.

Ex. 8. Given $\left.\begin{array}{l} 12:x::y:3, \\ \sqrt{x}+\sqrt{y}=5, \end{array}\right\}$ to find the values of x and y.

Ans. $x=9$ or 4, $y=4$ or 9.

Ex. 9. Given $\left.\begin{array}{l} (3x+4y)(7x-2y)+3x+4y=44, \\ (3x+4y)(7x-2y)-7x+2y=30, \end{array}\right\}$ to find the values of x and y.

Ans. $x=1$ or $1\frac{7}{17}$, $y=2$ or $-\frac{1}{17}$.

Ex. 10. Given $\left.\begin{array}{l} -x^2+6xy-9y^2+4x-12y=4, \\ x^2-2xy+3y^2-4x+5y=53, \end{array}\right\}$ to find the values of x and y.

Ans. $x=11$ or $-7\frac{1}{2}$, $y=3$ or $-3\frac{1}{6}$.

Ex. 11. Given $\left.\begin{array}{l} 2(x^2+y^2)(x+y)=15xy, \\ 4(x^4-y^4)(x^2-y^2)=45x^2y^2, \end{array}\right\}$ to find the values of x and y.

Ans. $x=2$ or 1, $y=1$ or 2.

Ex. 12. Given $\left.\begin{array}{l} (x^2-y^2)(x-y)=16xy, \\ (x^4-y^4)(x^2-y^2)=640x^2y^2, \end{array}\right\}$ to find the values of x and y.

Ans. $x=9$ or 3, $y=3$ or 9.

Ex. 13. Given $\left.\begin{array}{l} x(x+y+z)=27, \\ y(x+y+z)=18, \\ z(x+y+z)=36, \end{array}\right\}$ to find the values of x, y, and z.

Ans. $x=3$, $y=2$, $z=4$.

Ex. 14. Given $\left.\begin{array}{l} xy=z, \\ yz=v, \\ xv=a, \\ yv=bx, \end{array}\right\}$ to find the values of x, y, z, and v.

Ans. $x=\frac{\sqrt{a}}{\sqrt[3]{b}}$, $y=\sqrt[3]{b}$, $z=\sqrt{a}$, $v=\sqrt{a}\sqrt[3]{b}$.

Ex. 15. Given $\left.\begin{array}{l} xyz=105, \\ xyv=135, \\ xzv=189, \\ yzv=315, \end{array}\right\}$ to find the values of x, y, z, and v.

Ans. $x=3$, $y=5$, $z=7$, $v=9$.

Ex. 16. Given $\left.\begin{array}{l} x^2+\dfrac{x^4}{y^2}+y^2=84, \\ x+\dfrac{x^2}{y}+y=14, \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$, $y=2$ or 8.

Ex. 17. Given $\left.\begin{array}{l} \sqrt{y}-\sqrt{a-x}=\sqrt{y-x}, \\ 2\sqrt{y-x}+2\sqrt{a-x}=5\sqrt{a-x}, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\frac{4}{5}a$, $y=\frac{5}{4}a$.

Ex. 18. Given $\left.\begin{array}{l} x^3+xy^2=ay, \\ x^2y+y^3=bx, \end{array}\right\}$ to find the values of x and y.

Ans. $x=\sqrt{\dfrac{\sqrt{a^3b}}{a+b}}$, $y=\sqrt{\dfrac{\sqrt{ab^3}}{a+b}}$.

Ex. 19. Given $\left.\begin{array}{l} \sqrt{5\sqrt{x}+5\sqrt{y}}+\sqrt{x}+\sqrt{y}=10, \\ \sqrt{x^5}+\sqrt{y^5}=275, \end{array}\right\}$ to find the values of x and y.

Remark. Put $z^2=\sqrt{x}+\sqrt{y}$. Then, from Eq. 1, $z=\sqrt{5}$; that is, $\sqrt{x}+\sqrt{y}=5$. Next put $\sqrt{x}=\frac{5}{2}+v$, and $\sqrt{y}=\frac{5}{2}-v$. Substituting these values in Eq. 2, we find $v^2=\frac{1}{4}$, or $v=\pm\frac{1}{2}$.

Ans. $x=9$ or 4, or $\dfrac{-13\pm\sqrt{-51}}{2}$.

$y=4$ or 9, or $\dfrac{-13\mp\sqrt{-51}}{2}$.

Several of the following examples have imaginary or incommensurable roots which are not here given.

Ex. 20. Given $\left.\begin{array}{l} (x^2+y^2)xy=13090, \\ x+y=18, \end{array}\right\}$ to find the values of x and y.

Ans. $x=7$ or 11, $y=11$ or 7.

Ex. 21. Given $\left.\begin{array}{l} 5(x^2+y^2)+4xy=356, \\ x^2+y^2+x+y=62, \end{array}\right\}$ to find the values of x and y.

Ans. $x=4$ or 6, $y=6$ or 4.

Ex. 22. Given $\left.\begin{array}{l}(x^2+y^2)xy=300,\\ x^4+y^4\quad=337,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm4$, $y=\pm3$.

Ex. 23. Given $\left.\begin{array}{l}(x^2+y^2)(x^3+y^3)=455,\\ x+y\qquad=\ \ 5,\end{array}\right\}$ to find the values of x and y.

Ans. $x=3$ or 2, $y=2$ or 3.

Ex. 24. Given $\left.\begin{array}{l}\dfrac{x^2+xy+y^2}{x+y}=14,\\ \dfrac{x^2-xy+y^2}{x-y}=18,\end{array}\right\}$ to find the values of x and y.

Ans. $x=12$, $y=6$.

Ex. 25. Given $\left.\begin{array}{l}(x^2-xy+y^2)(x^2+y^2)\qquad=\ 91,\\ (x^2-xy+y^2)(x^2+xy+y^2)=133,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm3$ or ±2, $y=\pm2$ or ±3.

Ex. 26. Given $\left.\begin{array}{l}(x+y)xy\quad=\ 30,\\ (x^2+y^2)x^2y^2=468,\end{array}\right\}$ to find the values of x and y.

Ans. $x=2$ or 3, $y=3$ or 2.

Ex. 27. Given $\left.\begin{array}{l}x-y+\sqrt{\dfrac{x-y}{x+y}}=\dfrac{12}{x+y},\\ x^2+y^2=41,\end{array}\right\}$ to find the values of x and y.

Ans. $x=\pm5$, $y=\pm4$.

Ex. 28. Given $\left.\begin{array}{l}(x+y)^3+x+y=30,\\ x-y=\ 1,\end{array}\right\}$ to find the values of x and y.

Remark. Multiply Eq. 1 by $x+y$, and we have

$$(x+y)^4+(x+y)^2=30(x+y).$$

Add to each member $9(x+y)^2+25$, and the square root of each member of the equation may be extracted.

Ans. $x=2$, $y=1$.

Ex. 29. Given $\left.\begin{array}{l}(x+y)(xy+1)=\ 18xy\\ (x^2+y^2)(x^2y^2+1)=208x^2y^2\end{array}\right\}$ to find the values of x and y.

Remark. Divide Eq. 1 by xy, and Eq. 2 by x^2y^2, and we have

$$x+y+\frac{1}{x}+\frac{1}{y}=18.$$

$$x^2+y^2+\frac{1}{x^2}+\frac{1}{y^2}=208.$$

Put $x+\frac{1}{x}=z$, and $y+\frac{1}{y}=v$.

Then $z+v=18$, and $z^2+v^2=212$.

Whence $z=14$ or 4, and $v=4$ or 14; and hence x and y are easily found.

Ans. $x=2\pm\sqrt{3}$, $y=7\pm4\sqrt{3}$.

PROBLEMS INVOLVING EQUATIONS OF THE SECOND DEGREE WITH SEVERAL UNKNOWN QUANTITIES.

Prob. 1. If I increase the numerator of a certain fraction by 2, and diminish the denominator by 2, I obtain the reciprocal of the first fraction; also, if I diminish the numerator by 2, and increase the denominator by 2, the resulting fraction, increased by $1\frac{1}{15}$, is equal to the reciprocal of the first fraction. What is the fraction? *Ans.* $\frac{5}{7}$.

Prob. 2. It is required to divide the number 102 into three parts, such that the product of the first and third shall be equal to 102 times the second part, and the third part shall be $1\frac{1}{2}$ times the first.

Ans. The first part is 34, the second 17, and the third 51.

Prob. 3. A certain number consists of two digits. If I invert the digits, and multiply this new number by the first, I obtain for a product 5092; but if I divide the first digit by the second, I obtain 1 for a quotient with 1 for a remainder. What is the number? *Ans.* 76.

Prob. 4. The fore wheel of a carriage makes 165 more revolutions than the hind wheel in going 5775 feet; but if the circumference of each wheel be increased $2\frac{1}{2}$ feet, the fore wheel will make only 112 revolutions more than the hind wheel in the same space. Required the circumference of each wheel.

Ans. The fore wheel 10 feet, the hind wheel 14 feet.

Prob. 5. A piece of cloth, by being wet in water, shrinks one eighth in its length and one sixteenth in its breadth. If the perimeter of the piece is diminished $4\frac{1}{4}$ feet, and the surface $5\frac{3}{4}$ square feet, by wetting, what were the length and breadth of the piece?

Ans. 16 feet long and 2 feet wide.

Prob. 6. A certain number of laborers in 8 hours transport a pile of stones from one place to another. If there were 8 more laborers, and if each carried each time 5 pounds less, the pile would be removed in 7 hours; but if there were 8 less laborers, and if each carried each time 11 pounds more, it would require 9 hours to remove the pile. How many laborers were there employed, and how many pounds did each carry?

Ans. 28 laborers, and each carried 45 pounds; or 36 laborers, and each carried 77 pounds.

Prob. 7. A certain capital yields yearly $\$123\frac{1}{2}$ interest; a second capital, \$700 larger, and loaned at $\frac{1}{4}$ per cent. less, yields yearly $\$11\frac{1}{2}$ more interest than the first. How large was the first capital, and at what per cent. was it loaned?

Ans. The capital was \$3800, loaned at $3\frac{1}{4}$ per cent.

Prob. 8. A person bought a number of \$20 railway shares when they were at a certain rate per cent. discount for \$1500; and afterward, when they were at the same rate per cent. premium, sold them all but 60 for \$1000. How many did he buy, and what did he give for each of them?

Ans. 100 shares at \$15 each.

Prob. 9. A rectangular lot is 119 feet long and 19 feet broad. How much must be added to the breadth, and how much taken from the length, in order that the perimeter may be increased by 24 feet, and the contents of the lot remain the same?

Ans. The length must be diminished 102 feet, and the breadth increased 114 feet.

Prob. 10. There are two numbers such that their sum and product together amount to 47; also, the sum of their squares exceeds the sum of the numbers themselves by 62. What are the numbers?

Ans. 5 and 7.

Prob. 11. The sum of two numbers is a, and the sum of their reciprocals is b. Required the numbers.

Ans. $\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - \frac{a}{b}}$.

Prob. 12. A and B engage to reap a field for \$24; and as A alone could reap it in nine days, they promise to complete it in five days. They found, however, that they were obliged to call in C to assist them for the last two days, in consequence of which B received one dollar less than he otherwise would have done. In what time could B or C alone reap the field?

Ans. B in 15 and C in 18 days.

Prob. 13. The sum of the cubes of two numbers is 35, and the sum of their ninth powers is 20,195. Required the numbers.

Ans. 2 and 3.

Prob. 14. There are two numbers whose product is 300; and the difference of their cubes is thirty-seven times the cube of their difference. What are the numbers?

Ans. 20 and 15.

Prob. 15. A merchant had $26,000, which he divided into two parts, and placed them at interest in such a manner that the incomes from them were equal. If he had put out the first portion at the same rate as the second, he would have drawn for this part $720 interest; and if he had placed the second out at the same rate as the first, he would have drawn for it $980 interest. What were the two rates of interest?

Ans. 6 per cent. for the larger sum, and 7 for the smaller.

Prob. 16. A miner bought two cubical masses of ore for $820. Each of them cost as many dollars per cubic foot as there were feet in a side of the other; and the base of the greater contained a square yard more than the base of the less. What was the price of each? *Ans.* 500 and 320 dollars.

Prob. 17. A gentleman bought a rectangular lot of land at the rate of ten dollars for every foot in the perimeter. If the same quantity had been in a square form, and he had bought it at the same rate, it would have cost him $330 less; but if he had bought a square piece of the same perimeter, he would have had $12\frac{1}{4}$ rods more. What were the dimensions of the lot?

Ans. 9 by 16 rods.

Prob. 18. A and B put out at interest sums amounting to $2400. A's rate of interest was one per cent. more than B's; his yearly interest was five sixths of B's; and at the end of ten years his principal and simple interest amounted to five sevenths of B's. What sum was put at interest by each, and at what rate?

Ans. A $960 at 5 per cent., B $1440 at 4 per cent.

Prob. 19. A person bought a quantity of cloth of two sorts for $63. For every yard of the best piece he gave as many dollars as he had yards in all; and for every yard of the poor-year he sowed 1,048,576 pecks, by how many times must the seed have increased each harvest, supposing the increase to have been always the same? *Ans.* Four times.

Ex. 19. There are three numbers in geometrical progression, the difference of whose differences is six, and their sum is forty-two. Required the numbers. *Ans.* 6, 12, and 24.

Ex. 20. There are three numbers in geometrical progression, the greatest of which exceeds the least by 24; and the difference of the squares of the greatest and the least is to the sum of the squares of all the three numbers as 5 : 7. What are the numbers? *Ans.* 8, 16, and 32.

Ex. 21. There are three numbers in geometrical progression whose continued product is 216, and the sum of their cubes is 1971. Required the numbers. *Ans.* 3, 6, and 12.

Ex. 22. There are four numbers in geometrical progression whose sum is 350; and the difference between the extremes is to the difference of the means as 37 : 12. What are the numbers? *Ans.* 54, 72, 96, 128.

Ex. 23. There are four numbers, the first three of which are in geometrical progression, and the last three in arithmetical progression; the sum of the first and last is 14, and that of the second and third 12; find the numbers.

Ans. 2, 4, 8, 12; or $\frac{25}{2}, \frac{15}{2}, \frac{9}{2}, \frac{3}{2}$.

Ex. 24. Three numbers whose sum is 15 are in arithmetical progression; if 1, 4, and 19 be added to them respectively, they are in geometrical progression. Determine the numbers.

Ans. 2, 5, 8.

THE END.